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ON THE N-BODY PROBLEM

by

Zhifu Xie

A dissertation submitted to the faculty of

Brigham Young University

in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics

Brigham Young University

August 2006

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GRADUATE COMMITTEE APPROVAL

of a dissertation submitted by

Zhifu Xie

This dissertation has been read by each member of the following graduate committee and by majority vote has been found to be satisfactory.

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As chair of the candidate's graduate committee, I have read the dissertation of Zhifu Xie in its final form and have found that (1) its format, citations, and bibliographical style are consistent and acceptable and fulfill university and department style requirements; (2) its illustrative materials including figures, tables, and charts are in place; and (3) the final manuscript is satisfactory to the graduate committee and is ready for submission to the university library.

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ABSTRACT

ON THE N-BODY PROBLEM

Zhifu Xie

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Doctor of Philosophy

In this thesis, central configurations, regularization of simultaneous binary collision, linear stability of Kepler orbits, and index theory for symplectic path are studied. The history of their study is summarized in section 1.

Section 2 deals with the following problem: given a collinear configuration of 4 bodies, under what conditions is it possible to choose positive masses which make it central. It is always possible to choose three positive masses such that the given three positions with the masses form a central configuration. However, for an arbitrary configuration of 4 bodies, it is not always possible to find positive masses forming a central configuration. An expression of four masses is established depending on the position x and the center of mass u, which gives a central configuration in the collinear four body problem. Specifically it is proved that there is a compact region in which no central configuration is possible for positive masses. Conversely, for any configuration in the complement of the compact region, it is always possible to choose positive masses to make the configuration central.

The singularities of simultaneous binary collisions in collinear four-body problem is regularized by explicitly constructing new coordinates and time transformation in section 3. The motion in the new coordinates and time scale across simultaneous binary collision is at least C^2. Furthermore, the

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behavior of the motion closing, across and after the simultaneous binary collision, is also studied. Many different types of periodic solutions involving single binary collisions and simultaneous binary collisions are constructed.

In section 4, the linear stability is studied for the Kepler orbits of the rhombus four-body problem. We show that, for given four proper masses, there exists a family of periodic solutions for which each body with the proper mass is at the vertex of a rhombus and travels along an elliptic Kepler orbit. Instead of studying the 8 degrees of freedom Hamilton system for planar four-body problem, we reduce this number by means of some symmetry to derive a two degrees of freedom system which then can be used to determine the linear instability of the periodic solutions. After making a clever change of coordinates, a two dimensional ordinary differential equation system is obtained, which governs the linear instability of the periodic solutions. The system is surprisingly simple and depends only on the length of the sides of the rhombus and the eccentricity e of the Kepler orbit.

In section 5, index theory for symplectic paths introduced by Y. Long is applied to study the stability of a periodic solution x for a Hamiltonian system. We establish a necessary and sufficient condition for stability of the periodic solution x in two and four dimension.

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ACKNOWLEDGMENTS

First of all, I would like to take this opportunity to express my sincere gratitude to my advisor, Dr. Ouyang Tiancheng, for his tremendous amount of time, energy, and support he has given me during these years. I would like to thank all the committee members for reading my dissertation and giving me lots of valuable suggestion. In particular, I would like to thank Dr. Bakker Lennard for his time and effort in proofreading all my papers during these years at Brigham Young University. Finally, I am so grateful to all the math teachers I have had through these years, who lead me into the math field. I would also like to thank the department of mathematics at Brigham Young University for their unlimited supports.

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1 Introduction

Can we predict the motion of the Sun and the planets for the next billion years? We tend to think of the planets always moving in fixed orbits around the Sun. But do they really, and will they continue to do so for, say, the next billion years? Is the motion of the planets predictable and stable [6]? The French mathematician Pierre Simon de Laplace [15] said in 1843 that

> *"··· if we conceive of an intelligence that at a given instant comprehends all the relations of the entities of this universe, it could state the respective position, motions, and general affects of all these entities at any time in the past or future."*

But, since we are only able to approximate the current configuration of the Solar System, (i.e. the masses, positions, and velocities of all the objects in the Solar System), can we predict the future configuration of the Solar System by the Newtonian model? Will the Solar System in a billion years look like it does today? Is the Solar System stable? Over the past 350 years, many have attempted to answer these questions.

The first complete mathematical formulation of this problem appeared in Newton's Principia [25]. Since gravity was responsible for the motion of planets and stars, Newton had to express gravitational interactions in terms of differential equations. Consider N point particles with positive masses m_j and positions $q_j \in \mathbb{R}^d (j = 1, \cdots, N)$, where d is the dimension of the space where the particles live. The motion of the particles, which is governed by Newton's gravitational law, can be stated as

$$m_j \ddot{q}_j = \sum_{k \neq j}^{N} \frac{m_j m_k (q_k - q_j)}{|q_k - q_j|^3} = \frac{\partial U(q)}{\partial q_j}, \tag{1.1}$$

where $q = (q_1, \cdots, q_N)$ and $U(q)$ is the Newtonian potential,

$$U(q) = \sum_{1 \leq k < j \leq n} \frac{m_k m_j}{|q_k - q_j|}. \tag{1.2}$$

These equations are called the N-body problem of celestial mechanics.

There are many problems associated with the dynamics of such a system, for example, existence of periodic solutions, stability of periodic solutions, regularization of singularities, central configurations and so on. In the early stages of the study of N-body problem its main task was to calculate the orbits of the planets and predict the ephemerides over a long time. "However, the mathematical difficulties connected with this field inspired more and more the study of basic

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theoretical problems leading to the development of new mathematical tools" (Jürgen Moser, *Stable and Random Motions in Dynamical Systems with Special emphasis on Celestial Mechanics* [24]). The problem of stability, which concerns the behavior of the solutions for an infinite time interval, is one of the oldest questions in dynamical systems. Are there solutions which do not experience collisions and do not escape? Are there solutions stable? These questions can be answered by the construction of periodic solutions and quasi-periodic solutions. The successful construction of such solutions for N-body problem is due to Carl Ludwig Siegel, Andrei Nikolaevich Kolmogorov, Vladimir Igorevich Arnold and Jürgen Moser. It provides a set of positive measure of rigorous solutions which avoid collisions and infinity for all time [9], [24]. Instead of studying the basic theoretical problem, we study the properties of some specific solutions. In section 3 and section 4, we study the behavior of collision solution in collinear four-body problem and the problem of the stability of a Kepler solution.

The study of N-body problem has successfully developed differential and integral calculus, convergence of series expansions, and chaotic dynamics. Many natural questions in N-body problem are difficult to solve with current theory, especially as the value of N is increased [14]. Many great mathematicians, such as Leonard Euler (1707-1783), Louis Lagrange (1736-1813), G.G. Jacobi (1804-1851), George W. Hill (1838-1914), Henri Poincaré (1854-1912), George D. Birkhoff (1884-1944), Carl Ludwig Siegel (1896-1981), Andrei Nikolaevich Kolmogorov (1903-1987), Vladimir Igorevich Arnold (1937-), Jürgen Moser (1928-1999) and many others, attacked this problem on account of its importance for astronomy. "Despite the efforts by such outstanding mathematicians for over 200 years, the problem for $n > 2$ remains unsolved to this day" (C.L. Siegel and J.K. Moser, Page 20 in [43]). Newton once wrote that he only got headaches when he studied the three-body problem [50].

Contributions and Outline of the Thesis

1.1 Central Configurations

In order to make progress against such complexity, we can simplify questions by making assumptions about the parameters of the system. The most successful example is the study of a particular periodic orbit of the planar N-body problem in which the particles remain in the same shape rel-

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ative to one another [14]. The shapes possible for the particles in such orbits are called *central configurations* (this term appeared first in [51]). A configuration $q = (q_1, \cdots, q_n)$ is *collinear* if all the q_is are located on a line. A collinear central configuration is called a *Moulton configuration* after F.R. Moulton who proved that for a fixed mass vector $m = (m_1, \cdots, m_n)$ and a fixed ordering of the bodies along the line, there exists a unique collinear central configuration (up to translation and scaling) [23]. In section 2, we consider the inverse problem: given a collinear configuration, find the positive mass vectors, if any, for which it is a central configuration. Recall for any given three collinear positions, it is always possible to choose three positive masses making it central [1]. This result is generally not true for $n \geq 4$. The main result in section 2 is to prove that for $n = 4$, there exists a compact region E in configuration space such that within E it is not possible to choose a positive mass vector to make the configuration central. Furthermore, on the complement G of the compact region E, there always exists positive mass vector to make it central. The equations determining central configurations can be generalized to define them in higher dimensions as well; more precise definitions will be given in section 2. Results on this section and other related results are based upon those obtained in [7], [30], [31], [48].

1.2 Regularization of Singularity

A position $q = (q_1, \cdots, q_N)$ of the particles will be called a collision if $q_i = q_j$ for some $i \neq j$. Let $\triangle_{ij} := \{q \in \mathbb{R}^{Nd}, q_i = q_j\}$ and $\triangle := \bigcup_{1 \leq j < i \leq N} \triangle_{ij}$ be the set of collisions. The equations (1.1) are defined everywhere except at collisions. Suppose we are given the position and momentum of the particles at time $t = t_1$. If we do not start at a collision, then the standard theorems of differential equations assure the existence and uniqueness of a solution of equations (1.1) on some maximal interval $[t_1, t_2)$. If $t_2 < \infty$, then the solution is said to experience a singularity at t_2.

The behavior of a solution as it approaches a singularity is not fully understood, but some of the possibilities are known. If all of the particles approach a limiting position as $t \to t_2$, it is not difficult to show that the limiting position must be a collision [43], [51]. The singularity is then said to be due to collision and the solution is said to end in collision. If m of the particles coincide while the rest have distinct positions, then the collision is called an m-tuple collision.

For the two-body problem one can change variables so that a binary collision transforms to a regular point of the equations. Such a transformation is called a regularization of the binary

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collision. The solution can then be extended through the singularity. The extension corresponds physically to an elastic bounce. Since binary collision can be regularized, one is lead to ask whether the same can be done to other singularities. The singularity of collinear triple collision can not be regularized; see the work of R. McGehee [36]. The singularity of simultaneous binary collisions (SBC) is the only case left open for investigations. There are many research papers which studied the regularization of simultaneous binary collision with some assumption on masses; see the work of Belbruno [2], Punosevac and Wang [35], Simo and Lacomba [47] etc.

In section 3.1 we construct coordinate transforms in new time scale that remove the singularities of simultaneous binary collision in collinear four-body problem without any assumption on mass. The regularization is at least of class C^2. Based on the results of regularization of SBC in section 3.1, the behavior of the motion is studied for the motion acrossing collisions. The existence of a family of periodic solutions with simultaneous binary collision is proved in section 3.2. More periodic solutions involving single binary collision and SBC are constructed in section 3.3. Results on this section are based upon those obtained in [33].

1.3 Linear Stability of Homographic Periodic Solutions for Rhombus Shape

Planar central configurations give rise to a family of homographic periodic solutions. In section 4 we consider the linear stability of the Kepler orbits of the rhombus four-body problem. We show that, for given four proper masses, there exists a family of periodic solutions for which each body with the proper mass is at the vertex of a rhombus and travels along an elliptic Kepler orbit. Instead of studying the 8 degrees of freedom Hamilton system for planar four-body problem, we reduce this number by means of some symmetry to derive a two degrees of freedom system which then can be used to determine the linear instability of the periodic solutions. After making a clever change of coordinates, a two dimensional ordinary differential equation system is obtained, which governs the linear instability of the periodic solutions. The system is surprisingly simple and depends only on the length of the sides of the rhombus and the eccentricity e of the Kepler orbit. We prove the homographic periodic solutions of the rhombus four-body problem is linearly unstable. Results on this section are based upon those obtained in [32].

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1.4 Linear Stability and Index Theory for Symplectic Paths

For general periodic solution of N-body problem, it is very hard to reduce the dimension. We apply index theory for symplectic paths introduced by Y. Long to study the stability of a periodic solution x for a Hamiltonian system in section 5. Based on the study of the Y. Long's book [17], here we get some preliminary results on the relationship between stability and index theory. We establish a necessary and sufficient condition for stability of the periodic solution x in two dimension. We prove that the solution x is linear stable if and only if its index $ind(x)$ is an odd integer. We also get some important results in four dimension and they could be applied to study the linear stability of periodic solutions in some $N-$body problem. Results on this section are based upon those obtained in [34].

2 Central Configuration in Collinear Four-body Problem

2.1 Importance of Central Configurations

After making the definition of central configurations more precisely, this section will establish some notations and briefly sketch the important applications of central configurations.

In the N-body problem, we consider N particles at $q_i \in \mathbb{R}^d$ with positive masses $m_i \in \mathbb{R}^+$, $i = 1, \cdots, N$, and the dynamics given by equation (1.1), where U is the Newtonian potential defined in equation (1.2). We will use $q = (q_1, \cdots, q_N) \in \mathbb{R}^{Nd}$ and $m = (m_1, \cdots, m_N) \in \mathbb{R}^{N+}$ to denote the position and mass vectors respectively. Let

$$C = m_1 q_1 + \cdots + m_n q_N, \qquad M = m_1 + \cdots + m_N, \qquad c = C/M$$

be the first moment, total mass and center of mass of the bodies, respectively.

Definition 2.1. A configuration $q = (q_1, \cdots, q_N)$ is called a *central configuration* if the acceleration vectors of the bodies satisfy:

$$\sum_{j=1, j\neq k}^{N} \frac{m_j(q_j - q_k)}{|q_j - q_k|^3} = -\lambda(q_k - c) \qquad 1 \le k \le N. \tag{2.1}$$

for a constant λ.

The value of the constant λ in (2.1) is uniquely determined by

$$\lambda = \frac{U}{I}, \tag{2.2}$$

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where $I = \sum_{k=1}^{N} m_k |q_k|^2$. Moreover, if A is an orthogonal matrix, then clearly $Aq = (Aq_1, \cdots, Aq_N)$ is also a central configuration with the same λ. If a scalar $k \neq 0$, then $kq = (kq_1, \cdots, kq_N)$ is also a central configuration with λ replaced by $\frac{\lambda}{k^3}$. Thus any configuration similar to a central configuration, either by rescaling or rotating, is also a central configuration. When counting central configurations, we count only similarity classes.

A complete understanding of the nature of central configuration is of fundamental importance to the N-body problem of celestial mechanics: these configurations play an essential role in the global structures of the solutions of the n-body problem. More properties of central configuration can be found in [38],[42]. Following the presentation in [38], we give some examples of the properties of central configuration here.

What happens when the masses are released from a central configuration with zero initial velocity? All particles accelerate toward the origin in such a way that the configuration collapses homothetically resulting in a collision singularity. Simple collision orbits of this kind were the first explicitly known solutions of the 3-body problem [12]. These are not the only possible orbits which end in collision of all N particles, but it can be shown that for any such orbits, the configuration is asymptotically a central configuration.

A planar central configuration may give rise to a family of periodic solutions. The particles, after being released from the central configuration with initial velocities normal to their position vectors and with magnitudes proportional to their distances from the origin, traverse an elliptical orbit as in the Kepler problem. Moreover the configuration remains similar to the initial configuration throughout the motion,varying only in size.

The best known example is the equilateral triangle case which can be used to analyze the Sun-Jupiter-Trojan asteroid configurations [49]. If the velocities are sufficiently large, the orbits will be circular. As the velocities tend to zero, the ellipses become more and more eccentric and the periodic solutions approach the collision solutions described in the previous paragraph.

Central configuration also plays a role in the study of the topology of the energy and angular momentum levels of the planar n-body problem. Because the n-body problem is a Hamiltonian system, the total energy is a constant of motion and so orbits in phase space move on level sets of the Hamiltonian. Similarly, the angular momentum is conserved. As total energy and the angular momentum vary, the topology of these level sets changes. It turns out that bifurcations occur exactly at the level which contains the circular periodic orbits mentioned above.

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Central configurations are important in the study of the N-body problem. It is natural to inquire about the collisions of some subset of particles in the N-body problem. Central configurations turn out to be the limit configurations in collisions. More precisely, if N points in the N-body problem collide simultaneously at a finite time t_2 then the rescaled position vector $q^* = (t - t_2)^{2/3} q$ has as its limit a central configuration with the same mass vector [14]. In the construction of periodic solution with simultaneous binary collision, central configuration plays an important role. More detail will be given in section 3.2.

Finally, from a mathematical viewpoint, the problem of central configurations is one of possible solvable problems in the study of the dynamics of the N-body problem although itself is a very difficult problem. Almost every result in three-body dynamics is related to central configurations. For example, to understand the stability of the configurations to perturbations, it amounts to understanding the linearization of the dynamics about the central configurations. More detail will be given in section 4.

For these and other reasons central configurations have been studied extensively [38], [42]. But the understanding of central configurations is very difficult and still far from being complete.

A basic question on central configuration is about the finiteness of the number of central configuration:

> Given positive real numbers $m_1, \cdots, m_N$ as the masses in the n-body problem of celestial mechanics, is the number of central configuration finite?

The problem is in Wintner's book (1941) on celestial mechanics. In 1991, Steve Smale [44] described the finiteness problem as one of the eighteen great problems not solved in the 20th century. Also V.I. Arnold, on behalf of the International Mathematical Union, has written to a number of mathematicians with a suggestion that they describe some great problems for the next century [45]. Problem 6 in Arnold's report is the finiteness of the number of central configuration as a mathematical problem for the 21 century.

Results pertaining to how many central configurations exist have appeared over the time period from Euler, Lagrange, and Moulton to the present. For the 3-body problem there are five central configurations: three found by Lagrange, two by Euler. Moulton proved that for a fixed mass vector $\bar{m} = (m_1, \cdots, m_N)$ and a fixed ordering of the bodies along the line, there exists a unique collinear central configuration (up to translation and scaling) [23].

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The inverse problem of the finiteness of the number of central configuration is also studied by many mathematicians: Given a configuration, find the mass vectors, if any, for which it is a central configuration. Moulton also considered the inverse problem for the collinear case (if all the q_is are located on a line) in [23]. His results depend on whether N is even or odd. Albouy and Moeckel also study the inverse problem of collinear configuration of N bodies in [1]. They prove that for $N \leq 6$, each configuration determines a one-parameter family of masses (after normalization of the total mass) and the parameter is the center of mass when N is even and the square of the angular velocity of the corresponding circular periodic orbit when N is odd. For $N \geq 7$, it is still open. In their study, masses are allowed to be negative. In this section we consider the inverse problem: given a collinear configuration, find the positive mass vectors, if any, for which it is a central configuration.

2.2 Central Configuration in the Collinear Four-body Problem

If we let

$$a_{jk} = \frac{(q_k - q_j)}{|q_k - q_j|^3} \text{ if } j \neq k, \qquad a_{jj} = 0, \qquad m = (m_1, m_2, \cdots, m_n)^T,$$

$$A = (a_{jk})_{1 \leq j,k \leq n},$$

then the central configuration equations become

$$Am = -\lambda q + L \bigotimes \mu = \bar{b}, \qquad \text{where} \quad L = (1, 1, \cdots, 1) \in \mathbb{R}^n \tag{2.3}$$

for some constant $\mu \in \mathbb{R}^d$, where $\bigotimes$ denotes the tensor product, i.e. $L \bigotimes \mu = (\mu, \cdots, \mu)^T$. Comparing μ in (2.3) with c in (2.1), we have $\mu = \lambda c$.

Due to the fact that central configuration is invariant up to translation and rescaling, we can choose coordinates for the collinear four bodies as follows. Let $x_1 = -s - 1, x_2 = -1, x_3 = 1, x_4 = t + 1$, where $s, t > 0$. If we let $r = -\lambda, u = \mu$ (for ease of notation), then central configuration equation (2.1) in collinear four bodies case is

$$Am = rx + uL, \tag{2.4}$$

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where

$$A=\begin{bmatrix} 0 & s^{-2} & (s+2)^{-2} & (s+t+2)^{-2} \\ -s^{-2} & 0 & 1/4 & (t+2)^{-2} \\ -(s+2)^{-2} & -1/4 & 0 & t^{-2} \\ -(s+t+2)^{-2} & -(t+2)^{-2} & -t^{-2} & 0 \end{bmatrix}. \tag{2.5}$$

and $\bar{b} = rx + uL = (r(-s-1)+u, -r+u, r+u, r(t+1)+u)^T$.

In this section, the following results are obtained :

Theorem 2.1. *Let $x = (x_1, x_2, x_3, x_4) = (-s-1, -1, 1, t+1)$ be a collinear configuration with positive mass vector $m = (m_1, m_2, m_3, m_4)$ and assume that the center of mass $c = u/\lambda = \sum_{i=1}^{4} m_i x_i = 0$. Then there exist two constants t_0, s_0 and two implicit functions $pm_{02}(t,s) = 0$ and $pm_{03}(t,s) = 0$ which are defined by (2.8), (2.9) respectively, such that*

1) $pm_{02}(t,s) = pm_{03}(s,t)$.

2) The equation $pm_{02}(t,s) = 0$ can be globally solved for t to get a smooth monotone increasing function $t_2 = f(s_2)$. Furthermore, $\lim_{s_2 \to \infty} f(s_2) = \infty$. Similarly, we can get a smooth monotone increasing function $s_3 = f(t_3)$ from $pm_{03}(t,s) = 0$ such that $\lim_{t_3 \to \infty} f(t_3) = \infty$.

Then there exist an unbounded stripe-like region B bounded below by $s = s_0$, bounded to the left by $t = t_0$, between $(f(s_2), s_2)$ and $(t_3, f(t_3))$. For any point $(t,s) \in B$, with the center of mass at origin, the configuration x can be a central configuration with a positive mass $m = (m_1, m_2, m_3, m_4)$ which is unique up to a scalar.

Remark 1: Numerically, the region B is the one shown in figure 1. Here $t_0 = s_0 = 1.396812289$.

Theorem 2.2. *Let $x = (x_1, x_2, x_3, x_4) = (-s-1, -1, 1, t+1)$ be a collinear configuration with positive mass vector $m = (m_1, m_2, m_3, m_4)$. Then there exist an unbounded region G in the first quadrant $(t > 0, s > 0)$ bounded away from the origin by an implicit function $h(t,s) = 0$ defined by (2.12), such that for any $(t,s) \in G$, there exist positive masses (m_1, m_2, m_3, m_4) making x as a central configuration. Conversely for any (t,s) in $E = R^{2+} \setminus G$, there is no positive mass $m = (m_1, m_2, m_3, m_4)$ making $x = (-s-1, -1, 1, t+1)$ central, where R^{2+} is the first quadrant in the plane.*

Remark 2: Numerically the region G is the one shown in figure 2.

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The mass vector $m = (m_1, m_2, m_3, m_4) = m(x, u)$ depends on the position and center of mass up to a scalar. For fixed x, u, there is a unique solution $m(x, u)$ making x central with center of mass

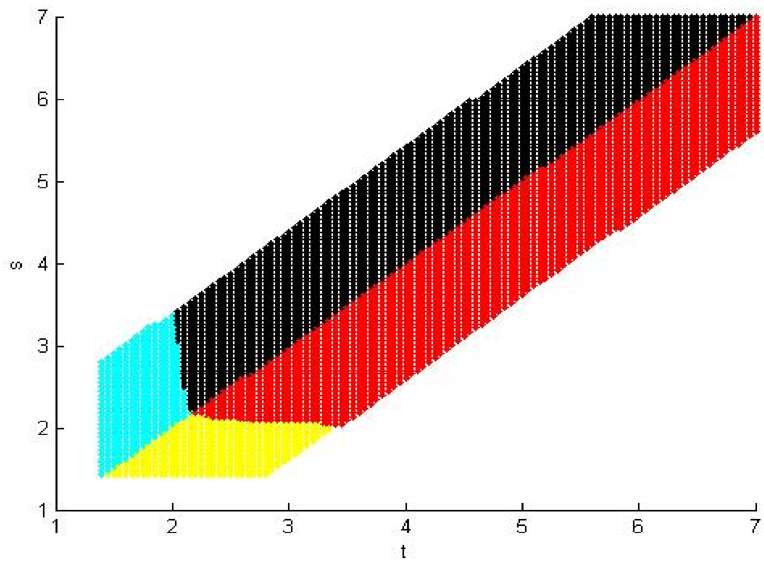

Figure 1: Central Configuration Region with Center at Origin

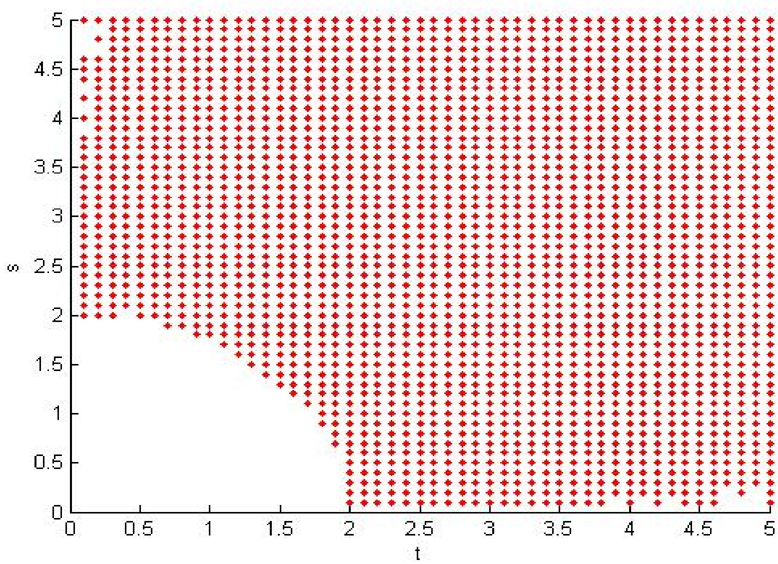

Figure 2: Central Configuration Region

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at u. Define

$\underline{u}(x) := \underline{u}(t,s) := \min\{u|x$ forms a central configuration centered at u with positive mass m(x,u)$\}$,

$\bar{u}(x) := \bar{u}(t,s) := \max\{u|x$ forms a central configuration centered at u with positive mass m(x,u)$\}$.

If the set $\{u|x$ forms a central configuration centered at u with positive mass m(x,u)$\}$ is empty, let $\underline{u}(x) = \bar{u}(x) = 0$. Defining

$$d(t,s) := \bar{u}(t,s) - \underline{u}(t,s),$$

we have the following.

Theorem 2.3. *1) For each point $(t_0, s_0) \in E$, $d(t_0, s_0) = 0$.*

2) For each point $(t_0, s_0) \in G$, $d(t_0, s_0) > 0$ and

$$\lim_{t_0 \to \infty, s_0 \to 0} d(t_0, s_0) = 0,$$

$$\lim_{t_0 \to 0, s_0 \to \infty} d(t_0, s_0) = 0.$$

2.2.1 General Solutions for 4-body Collinear Central Configuration

Given $s, t > 0$, we will find the general solution of masses $m_1, \cdots, m_4$ with two parameters r, u for the 4-body collinear central configuration, i.e. a solution of (2.4).

Because the matrix A defined by (2.5) is skew symmetric, the determinant of A is the square of its Pfaffian, that is $det(A) = (PfA)^2 = [a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}]^2 = (st)^{-2} - ((s+2)(t+2))^{-2} + \frac{1}{4}(s+t+2)^{-2} > 0$. So the matrix has full rank. Therefore, the solution is uniquely determined. Albouy and Mockel [1] proved that the given 4-body collinear configuration determines a two-parameter family of masses making it central but allowing masses to be negative. Here we can find a solution of masses explicitly by standard row reduction:

$$m_3 = \left(\left(-\frac{r(t+1)+u}{s^2} + \frac{-r+u}{(s+t+2)^2}\right)s^{-2} - \frac{r(-s-1)+u}{s^2(t+2)^2}\right)$$

$$\left(\left(\frac{1}{s^2t^2} + 1/4\,(s+t+2)^{-2}\right)s^{-2} - \frac{1}{s^2(t+2)^2(s+2)^2}\right)^{-1},$$

$$m_4 = \left(\left(-\frac{r+u}{s^2} + \frac{-r+u}{(s+2)^2}\right)s^{-2} - 1/4\,\frac{r(-s-1)+u}{s^2}\right)$$

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$$\left(\left(-\frac{1}{s^2t^2}+\frac{1}{(s+2)^2(t+2)^2}\right)s^{-2}-1/4\,\frac{1}{s^2(s+t+2)^2}\right)^{-1}.$$

If we write the central configuration equation from right to left, i.e. from m_4 to m_1, we have $m'=(m_4,m_3,m_2,m_1)$, $x'=(x_4,x_3,x_2,x_1)$, r,u the same constants, then the coefficient matrix is

$$B=\begin{bmatrix} 0 & -t^{-2} & -(t+2)^{-2} & -(s+t+2)^{-2} \\ t^{-2} & 0 & -1/4 & -(s+2)^{-2} \\ (t+2)^{-2} & 1/4 & 0 & -s^{-2} \\ (s+t+2)^{-2} & (s+2)^{-2} & s^{-2} & 0 \end{bmatrix}.$$

The central configuration equation changes to

$$Bm'=rx'+uL=(r(t+1)+u,r+u,-r+u,r(-s-1)+u)^T. \tag{2.6}$$

If the both sides of (2.6) are multiply by -1, s and t are exchanged, and u is switched to $-u$, then equation (2.6) will be the same as the equation (2.4). Therefore, m_1,m_2 are symmetrical to m_3,m_4 respectively in the sense that m_1 is equal to m_4 by exchanging s and t, and switching u to $-u$ in m_4 (similarly for m_2 and m_3). So m_1,m_2 have the following expressions:

$$m_1=\left(\left(-\frac{r-u}{t^2}+\frac{-r-u}{(t+2)^2}\right)t^{-2}-1/4\,\frac{r(-t-1)-u}{t^2}\right)$$

$$\left(\left(-\frac{1}{s^2t^2}+\frac{1}{(s+2)^2(t+2)^2}\right)t^{-2}-1/4\,\frac{1}{t^2(s+t+2)^2}\right)^{-1},$$

$$m_2=\left(\left(-\frac{r(s+1)-u}{t^2}+\frac{-r-u}{(s+t+2)^2}\right)t^{-2}-\frac{r(-t-1)-u}{t^2(s+2)^2}\right)$$

$$\left(\left(\frac{1}{s^2t^2}+1/4\,(s+t+2)^{-2}\right)t^{-2}-\frac{1}{(t+2)^2t^2(s+2)^2}\right)^{-1}.$$

Note that $r=-\lambda$ is negative if $m=(m_1,m_2,m_3,m_4)$ is a positive solution of (2.4). Also $m/|r|$ is a positive solution of

$$Am=-x+(u/|r|)L. \tag{2.7}$$

Because we are only concerned with the sign of the mass functions, we can assume $r=-1$. Under this assumption $u=\lambda c=-rc$ becomes the center of mass. In the following two sections, we will analyze the mass functions and find the possible region in $ts-$plane such that the mass functions are positive.

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2.2.2 Symmetrical Collinear Central Configuration

Again, in this subsection, we will fix the center of mass at the origin (i.e. $u = 0$) and let $r = -1$. Then x_2 is symmetric to x_3. By using symbolic computation of Matlab, we find the numerators and denominators of the masses. The numerators of masses are:

$$nm_1 = (s+2)^2 (s+t+2)^2 s^2 \left(-4\,t^2 - 16\,t - 16 + t^5 + 5\,t^4 + 8\,t^3\right),$$

$$nm_2 = 4\,(t+2)^2 \left(16 + 36\,s^2 t + 48\,s + 32\,s^3 + s^5 + 9\,s^4 + 40\,st + 8\,st^2 + 2\,s^4 t + 14\,s^3 t + \right.$$
$$\left. s^3 t^2 + 5\,s^2 t^2 + 56\,s^2 + 4\,t^2 + 16\,t - t^3 s^2 - 2\,t^4 s - 6\,t^3 s - t^5 - 5\,t^4 - 8\,t^3\right) s^2,$$

$$nm_3 = -4\left(-16 - 8\,s^2 t - 32\,t^3 - t^5 - 9\,t^4 - t^3 s^2 - 16\,s - 40\,st - 36\,st^2 - 5\,s^2 t^2 - 2\,t^4 s - \right.$$
$$\left. 14\,t^3 s - 4\,s^2 - 56\,t^2 - 48\,t + s^5 + 2\,s^4 t + 5\,s^4 + s^3 t^2 + 6\,s^3 t + 8\,s^3\right) t^2 (s+2)^2,$$

$$nm_4 = (t+2)^2 (s+t+2)^2 t^2 \left(-4\,s^2 - 16\,s - 16 + s^5 + 5\,s^4 + 8\,s^3\right).$$

They have the same positive denominator

$$dm_0 = 256 + 512\,t + 384\,t^2 + 16\,t^4 + 128\,t^3 + 384\,s^2 + 16\,s^4 + 128\,s^3 + 512\,s + 4\,s^4 t^2 + s^4 t^4 +$$
$$16\,t^3 s^3 + 4\,t^3 s^4 + 896\,st + 576\,st^2 + 16\,s^4 t + 160\,s^3 t + 64\,s^3 t^2 + 304\,s^2 t^2 + 16\,t^4 s +$$
$$160\,t^3 s + 4\,s^3 t^4 + 64\,t^3 s^2 + 4\,s^2 t^4 + 576\,s^2 t.$$

Then $m_i = \frac{nm_i}{dm_0}, 1 \le i \le 4$. Because the denominator is bigger than 256 for $s, t > 0$, the masses can not go to infinity if s, t are bounded. The configuration fails to be a central configuration if only if some of masses become negative. The possible negative terms in numerators are :

$$pm_{01} = t^5 + 5\,t^4 + 8\,t^3 - 4\,t^2 - 16\,t - 16,$$

$$pm_{02} = 16 + 48\,s + 56\,s^2 + 32\,s^3 + 9\,s^4 + s^5 + \left(16 + 40\,s + 36\,s^2 + 14\,s^3 + 2\,s^4\right) t +$$
$$\left(4 + 8\,s + 5\,s^2 + s^3\right) t^2 + \left(-8 - 6\,s - s^2\right) t^3 + (-5 - 2\,s)\, t^4 - t^5, \tag{2.8}$$

$$pm_{03} = 16 + 48\,t + 56\,t^2 + 32\,t^3 + 9\,t^4 + t^5 + \left(16 + 40\,t + 36\,t^2 + 14\,t^3 + 2\,t^4\right) s +$$
$$\left(4 + 8\,t + 5\,t^2 + t^3\right) s^2 + \left(-8 - 6\,t - t^2\right) s^3 + (-5 - 2\,t)\, s^4 - s^5, \tag{2.9}$$

$$pm_{04} = s^5 + 5\,s^4 + 8\,s^3 - 4\,s^2 - 16\,s - 16.$$

It is clear that the sign of the coefficients in the polynomial pm_{01} changes only once. By Descartes' rule of sign, there is exactly one positive root t_0 of pm_{01}. Because pm_{01} does not

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depend on s, the equation $pm_{01} = 0$ implicitly gives rise to a straight line $t = t_0$ in the ts-plane (Figure 3, $m_1 = 0$). Also m_1 is positive on the right of this line because pm_{01} goes to infinity as t goes to infinity. Similarly, the equation pm_{04} implicitly gives rise to a straight line $s = s_0$ in the ts-plane (Figure 3, $m_4 = 0$). m_4 is positive above this line because pm_{04} goes to infinity as s goes to infinity. So the region of $m_1 > 0$ and $m_4 > 0$ is a nonempty open set indicated in figure 3.

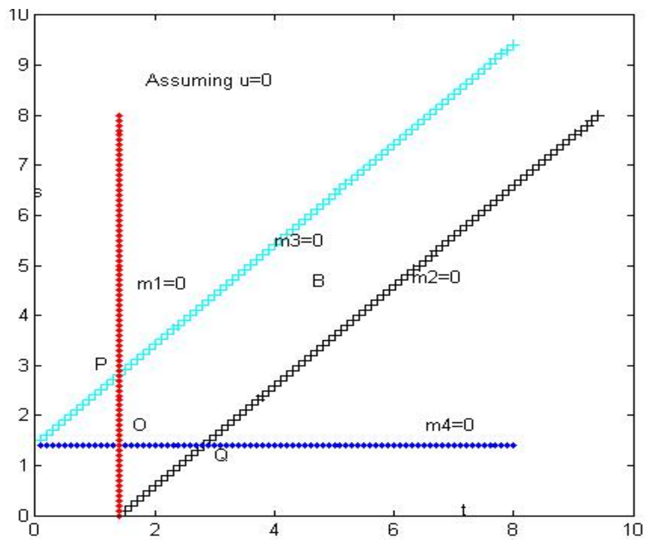

Figure 3: Boundary of C.C. Region

If we consider pm_{02} to be a polynomial in the variable t, then the coefficients are $c_0(s) = 16 + 48\,s + 56\,s^2 + 32\,s^3 + 9\,s^4 + s^5$, $c_1(s) = 16 + 40\,s + 36\,s^2 + 14\,s^3 + 2\,s^4$, $c_2(s) = 4 + 8\,s + 5\,s^2 + s^3$, $c_3(s) = -8 - 6\,s - s^2$, $c_4(s) = -5 - 2s$, $c_5(s) = -1$ with respect to the increasing order of the variable t, i.e. $pm_{02} = c_0(s) + c_1(s)t + c_2(s)t^2 + c_3(s)t^3 + c_4(s)t^4 + c_5(s)t^5$. For $s > 0$, c_0, c_1, c_2 are positive and c_3, c_4, c_5 are negative. So the sign of the polynomial pm_{02} changes only once implying there is exactly one positive root t for any given $s > 0$. Therefore, $pm_{02} = 0$ implicitly determines a smooth monotone increasing function $t = f(s)$ with the domain $s > 0$. Because the degree of the positive coefficients $c_0(s), c_1(s), c_2(s)$ are larger than the degree of $c_3(s), c_4(s), c_5(s)$, $t = f(s)$ must go to infinity as s goes to infinity. Similarly, $pm_{03} = 0$ implicitly determines a smooth monotone increasing function $s = g(t)$ with the domain $t > 0$. Moreover, the functions f and g are the same by the symmetry of s, t in pm_{02} and pm_{03}.

Now, we want to show that the implicit curves $pm_{02} = 0$ and $pm_{03} = 0$ have no intersecting

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points. That means the system of equations

$$\begin{cases} pm_{02} = 0 \\ pm_{03} = 0 \\ pm_{02} = pm_{03} \end{cases} \tag{2.10}$$

has no solution. Considering the difference of pm_{02} and pm_{03}, we have

$$pm_{02} - pm_{03} = 2\,(t-s)\,\big(t^4 + 7\,t^3 + 3\,st^3 + 20\,t^2 + 4\,s^2t^2 + 17\,st^2 +$$

$$3\,s^3t + 26\,t + 34\,st + 17\,s^2t + 16 + 20\,s^2 + s^4 + 7\,s^3 + 26\,s\big)$$

Then for $s, t > 0$, $pm_{02} - pm_{03} = 0$ if only if $s = t$. But for $s = t > 0$, $pm_{02} = pm_{03} = (16 + 64\,s + 100\,s^2 + 68\,s^3 + 17\,s^4)$ which has no zeros. So $pm_{02} = 0$ and $pm_{03} = 0$ can not be satisfied simultaneously, i.e. the two curves given by the two implicit functions have no intersecting points. Furthermore, the curve $(t, f(t))$ is above the curve $(f(s), s)$. Therefore, the four implicit curves give rise to a region (Figure 3, B) described in theorem 2.1. The region bounded by the four implicit curves in first quadrant is called *central configuration region.* For any point in the central configuration region, there are four unique positive masses which make it central with the center of mass at origin. (Note, the uniqueness is not true if the center of mass is not fixed. The mass vector will admit one parameter u). This completes the proof of Theorem 2.1.♯

Our investigation will now go into the change of the masses with respect to the positions. The intersecting point O of the line $m_1 = 0$ and $m_4 = 0$ is $(1.396812289, 1.396812289)$. The intersecting point P of the line $m_1 = 0$ and the curve $m_3 = 0$ is $(1.396812289, 2.807744118)$ with the same first coordinate as O. By symmetry, the point Q is $(2.807744118, 1.396812289)$. For example, the point $(1, 1)$ is out of the central configuration region for fixing center of mass at origin. It gives the configuration $x = (-2, -1, 1, 2)$. By solving equation (2.4) with $r = -1, u = 0$, the unique solution is $[-\frac{3168}{5201}, \frac{9540}{5201}, \frac{9540}{5201}, -\frac{3168}{5201}]$, which is not positive. That is the configuration $x = (-2, -1, 1, 2)$ could not be a central configuration by fixing the center of mass at origin. We will also show in the next section that the configuration $x = (-2, -1, 1, 2)$ could not be a central configuration even without fixing the center of mass.

Note that the central configuration region is symmetric along $s = t$. If $s = t$, we have $m_1 = m_4$ and $m_2 = m_3$. The different colors in figure 1 show that the heaviest mass of the four bodies

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changes as the change of the point in central configuration region. For example, when the point is in the triangle $\triangle POQ$ with $s > t$, m_3 becomes the heaviest mass. As the point moves up while t becomes larger with $s > t$, the heaviest mass m_3 decreases and will equal the mass m_1 on some curve, and eventually m_1 will become the heaviest mass.

We may intuitively think of m_1, m_4 as going to zero and m_2, m_3 going to infinity while $s = t$ goes to infinity because the outer two bodies travel much further than the inner two bodies do. Hence the smaller the masses of the outer two bodies the faster they travel. However, this is not the case. Both m_1, m_4 go to infinity as $s = t$ goes to infinity, but the limit of m_2, m_3 goes to a finite number 68 which is unexpected. Here we fix the center at the origin and get the symmetric region in which the central configuration of collinear four bodies lies. It is natural to ask how the center of mass affects the central configuration region. We will answer this question in the next subsection.

2.2.3 Proof of Theorem 2.2

In this subsection, we will find the central configuration region without fixing the center of mass in advance. For the reason previously given at the end of subsection 2.2.1, we can let $r = -1$. By using symbolic computation of Matlab, we find the numerators and denominators of the masses. They have the same positive denominators as given by

$$dm = 256 + 512\,s + 512\,t + s^4t^4 + 384\,s^2 + 16\,s^4 + 128\,s^3 + 384\,t^2 + 16\,t^4 + 128\,t^3 + 64\,s^3t^2$$
$$+64\,s^2t^3 + 4\,s^4t^3 + 4\,s^4t^2 + 4\,s^3t^4 + 16\,s^3t^3 + 4\,s^2t^4 + 576\,s^2t + 576\,st^2 + 896\,st +$$
$$16\,s^4t + 160\,s^3t + 16\,st^4 + 160\,st^3 + 304\,s^2t^2.$$

Furthermore, we find that the possible negative terms in each mass solution are the following.

$$pm_1(t,u) = -16\,u - 16 + (-16\,u - 16)\,t + (-4 - 4\,u)\,t^2 + (-4\,u + 8)\,t^3 + (-u + 5)\,t^4 + t^5,$$

$$pm_2(t,s,u) = 16 + 16\,u + 48\,s + us^4 + 56\,s^2 + s^5 + 32\,s^3 + 24\,us^2 + 32\,us + 8\,us^3 + 9\,s^4 +$$
$$\left(16 + 40\,s + 2\,us^3 + 24\,us + 36\,s^2 + 2\,s^4 + 14\,s^3 + 12\,us^2 + 16\,u\right)t +$$
$$\left(us^2 + 4 + 4\,us + 5\,s^2 + 4\,u + 8\,s + s^3\right)t^2 +$$
$$\left(-s^2 - 6\,s + 2\,us + 4\,u - 8\right)t^3 + (-5 + u - 2\,s)\,t^4 - t^5,$$

$$pm_3(t,s,u) = 16 - 16\,u - 24\,ut^2 + 48\,t + t^5 + 9\,t^4 + 32\,t^3 - t^4u - 32\,ut - 8\,t^3u + 56\,t^2 +$$

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$$\left(-2\,t^3u+14\,t^3-12\,ut^2-24\,ut+16+40\,t+36\,t^2+2\,t^4-16\,u\right)s+$$

$$\left(-ut^2+t^3-4\,ut+5\,t^2+8\,t+4-4\,u\right)s^2+$$

$$\left(-t^2-4\,u-8-6\,t-2\,ut\right)s^3+\left(-u-5-2\,t\right)s^4-s^5,$$

$$pm_4(s,u)=16\,u-16+\left(16\,u-16\right)s+\left(-4+4\,u\right)s^2+\left(4\,u+8\right)s^3+\left(u+5\right)s^4+s^5.$$

Let $k_1(t)=(t+2)^2, k_2(t,s)=(t+s+2)^2$. The mass solutions of (2.4) are

$$\begin{aligned}
m_1(t,s,u) &= \tfrac{k_1(s)k_2(t,s)s^2pm_1(t,u)}{dm},\\
m_2(t,s,u) &= \tfrac{4k_1(t)s^2pm_2(t,s,u)}{dm},\\
m_3(t,s,u) &= \tfrac{4k_1(s)t^2pm_3(t,s,u)}{dm},\\
m_4(t,s,u) &= \tfrac{k_1(t)k_2(t,s)t^2pm_4(s,u)}{dm},
\end{aligned} \tag{2.11}$$

which have the following relations:

$$m_1(t,s,u)=m_4(s,t,-u), \qquad m_2(t,s,u)=m_3(s,t,-u).$$

Then the central configuration region is the region on which $pm_1,\cdots,pm_4$ are all positive.We prove Theorem 2.2 by using the following Lemmas.

Lemma 2.1 The region in which $m_1>0, m_4>0$ for $s>0, t>0$ by choosing the proper u is the infinite region G in figure 6 bounded by an implicit function $h(t,s)=0$ far away from the origin.

Proof. Because pm_1 is independent on s, the positivity of pm_1 only depends on u and t. For each fixed u, the number of sign changes of the coefficients of pm_1 is at most one. More precisely, if $u>-1$, the sign of polynomial pm_1 changes only once. If $u<-1$, the coefficients of pm_1 are all positive. Therefore, pm_1 is always positive for $t>0$ while $u<-1$. When $u=-1$, we have $pm_1=12\,t^3+6\,t^4+t^5$ which is zero at $t=0$ and positive for $t>0$. By Descartes' rule of sign, there is exactly one positive root for any given $u>-1$. The equation $pm_1=0$ implicitly defines t as a function of u on $u>-1$. The curve is a smooth monotonically increasing curve by the property of polynomial functions as shown in figure 4.

Another way show this by considering the sign of its first derivative. From $pm_1=0$, it is easy to solve for u, which is

$$u=\frac{-4\,t^2-16\,t-16+8\,t^3+t^5+5\,t^4}{16\,t+16+t^4+4\,t^3+4\,t^2}$$

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$$\frac{du}{dt} = \frac{t^2\left(768\,t + 112\,t^3 + 576 + 416\,t^2 + t^6 + 8\,t^5 + 24\,t^4\right)}{\left(16\,t + 16 + t^4 + 4\,t^3 + 4\,t^2\right)^2}$$

which is always positive for $t > 0$.

Because u is the center of mass, u is a real value between $x_1 = -s - 1$ and $x_4 = t + 1$. In this graph, m_1 is positive for the points above the curve. For example, for $u = 0$, $t \geq 1.396812289$. For the same reasons, the implicit function $pm_4 = 0$ has similar properties and the implicit graph is given by figure 5.

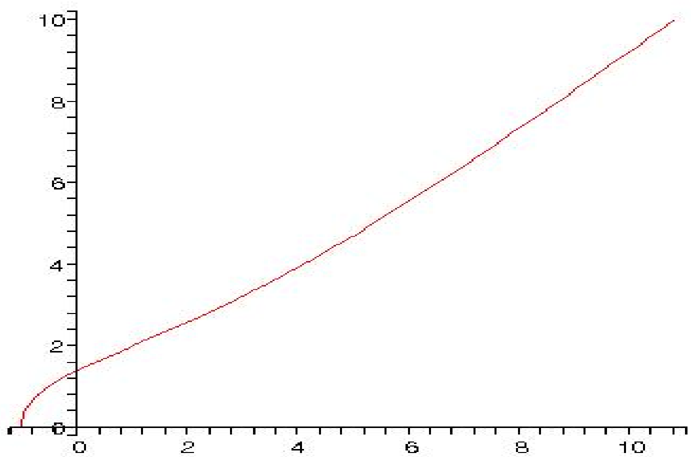

Figure 4: $m_1 = 0$

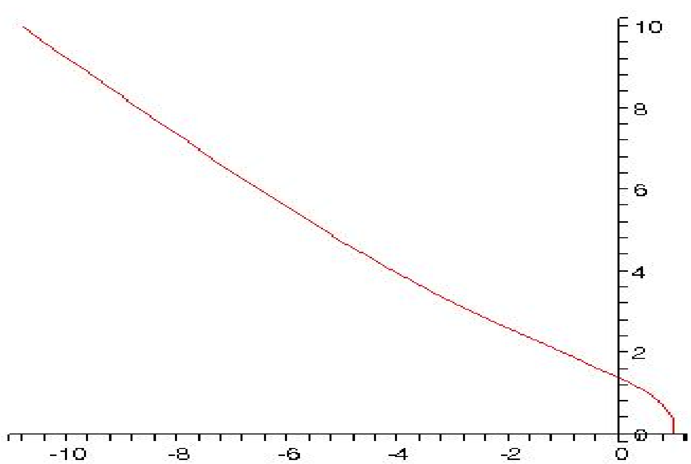

Figure 5: $m_4 = 0$

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Therefore, the two equations $pm_1 = 0$ and $pm_4 = 0$ give us an implicit function of (t, s) by eliminating u.

$$\frac{-4\,t^2 - 16\,t - 16 + 8\,t^3 + t^5 + 5\,t^4}{16 + 16\,t + t^4 + 4\,t^3 + 4\,t^2} + \frac{-4\,s^2 - 16\,s - 16 + 8\,s^3 + s^5 + 5\,s^4}{16 + 16\,s + s^4 + 4\,s^3 + 4\,s^2} = 0$$

Because its denominator is always positive for positive s, t, the equation is equivalent to $h(t, s) = 0$ which defines an implicit function, where

$$\begin{aligned} h(t,s) &= (-512 - 512\,t - 128\,t^2 + 64\,t^3 + 64\,t^4 + 16\,t^5) + \big((-512 - 512\,t - 128\,t^2 + 64\,t^3 \\ &+64\,t^4 + 16\,t^5)\big)\,s + \big((-128 - 128\,t - 32\,t^2 + 16\,t^3 + 16\,t^4 + 4\,t^5)\big)\,s^2 + \big((64 + 64\,t + 16\,t^2 \\ &+64\,t^3 + 28\,t^4 + 4\,t^5)\big)\,s^3 + \big((64 + 64\,t + 16\,t^2 + 28\,t^3 + 10\,t^4 + t^5)\big)\,s^4 \\ &+ \big((16 + 16\,t + 4\,t^2 + 4\,t^3 + t^4)\big)\,s^5 \end{aligned} \tag{2.12}$$

The function is symmetric on s, t, i.e. $h(t, s) = h(s, t)$. Note that

$$(-512 - 512\,t - 128\,t^2 + 64\,t^3 + 64\,t^4 + 16\,t^5) =$$

$$= 4(-128 - 128\,t - 32\,t^2 + 16\,t^3 + 16\,t^4 + 4\,t^5) = 16\,(t-2)\left(t^2 + 2\,t + 4\right)(t+2)^2$$

The coefficients of s^3, s^4, s^5 are all positive. By Descartes' rule, for $t \in (0, 2)$, the equation $h(t, s) = 0$ gives rise to an implicit curve Γ. But for $t \in (2, \infty)$, equation has no positive solutions because all terms in $h(t, s)$ are positive. The implicit curve Γ of (t, s) is the curve in figure 6 which divide the first quadrant into two parts, the region E and the unbounded region G.

Given any point $(t, s) \in \Gamma$, there is a unique u such that $pm_1 = pm_4 = 0$ because the implicit curve $pm_1 = 0$ is smooth and monotonically increasing. Now for any point $(t_0, s_0) \in G$, there exists $(t, s) \in \Gamma$ such that $t < t_0, s < s_0$. We can choose u to make $pm_1(t, u) = pm_4(s, u) = 0$. Therefore $pm_1(t_0, u) > 0$ and $pm_4(s_0, u) > 0$. At any point (t_0, s_0) in open region G, pm_1, pm_4 could be positive simultaneously for some u. By a similar argument, we can show that pm_1, pm_4 could not be positive simultaneously in the region E for any u. This completes the proof of Lemma 1.

□

Remark 3: We give here an example of the previous lemma. Let the configuration $x = (-2, -1, 1, 2)$ correspond to $(t, s) = (1, 1) \in E$. In this case, we get $m = [-\frac{3168}{5201} - \frac{5904}{5201}\,u, \frac{9540}{5201} +$

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$\frac{5436}{5201}u, \frac{9540}{5201} - \frac{5436}{5201}u, -\frac{3168}{5201} + \frac{5904}{5201}u]$ by solving equation (2.4) under $r = -1$. Note that $m_1 > 0$ implies $u < 0$ while $m_4 > 0$ implies $u > 0$. Lemma 1 tells us the region E is not a central configuration region but that region G could be. Here we restrict our regions in the first quadrant of ts-plane. In fact, we will show that the open region G is a central configuration region. We achieve this by use of the following lemmas.

Lemma 2.2 The implicit curve $pm_2 = 0$ intersects the vertical line $pm_1 = 0$ only at $s = 0$ for

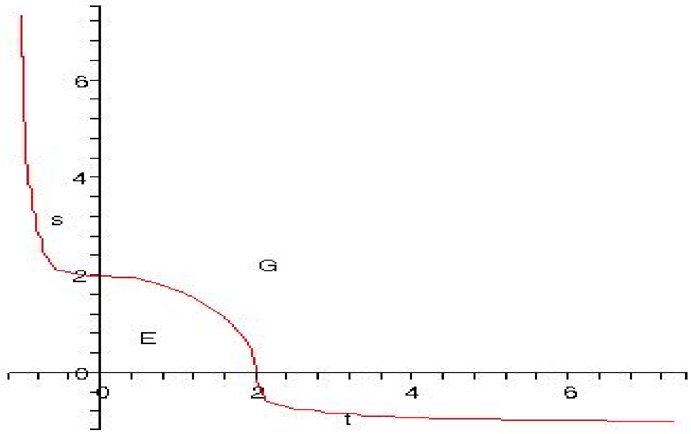

Figure 6: C.C. Region

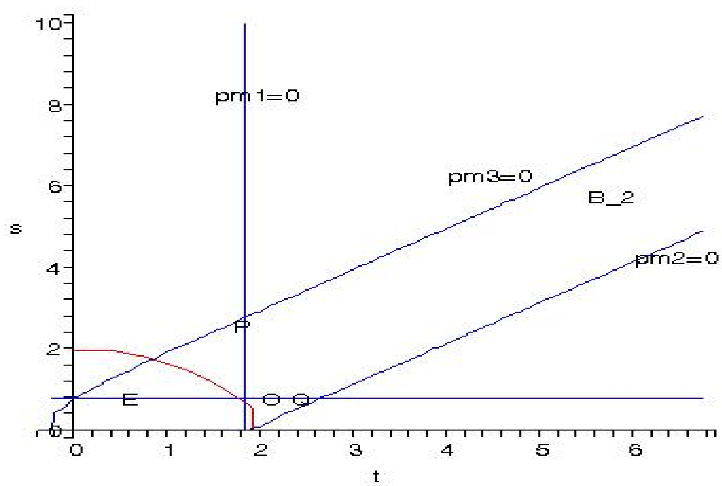

Figure 7: C.C. Region for fixed $-1 \leq u \leq 1$

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any $u \geq -1$ in the first quadrant of ts-plane. The implicit curve $pm_3 = 0$ intersects the horizontal line $pm_4 = 0$ only at $t = 0$ for any $u \leq -1$ in the first quadrant of ts-plane.

proof. In the proof of Lemma 2.1, we know that $pm_1 = 0$ gives rise to a vertical line for $u > -1$ and $pm_1 > 0$ for $u < -1$. $pm_1 = 0$ gives rise to the s-axis when $u = -1$. Solving for u from $pm_1 = 0$, we have

$$u = \frac{-4\,t^2 - 16\,t - 16 + 8\,t^3 + t^5 + 5\,t^4}{16 + 16\,t + t^4 + 4\,t^3 + 4\,t^2}. \tag{2.13}$$

Substituting u into $pm_2 = 0$ and simplifying, we have

$$\frac{s\left(256 + 16\,s^4 t + \cdots + 144\,st^5\right)}{16 + 16\,t + t^4 + 4\,t^3 + 4\,t^2} = 0. \tag{2.14}$$

The only solution is $s = 0$. So $pm_2 = 0$ intersects $pm_1 = 0$ only at $s = 0$ for any $u > -1$. We complete the proof of the first part. We can similarly prove the second part.

Lemma 2.3 Given any $u \leq -1$, the three equations $pm_2 = 0, pm_3 = 0, pm_4 = 0$ give rise to three implicit curves and the three implicit curves enclose a nonempty open central configuration region B_1 (shown in Figure 8). Given any $-1 \leq u \leq 1$, the four equations $pm_1 = 0, pm_2 = 0, pm_3 = 0, pm_4 = 0$ give rise to four implicit curves and the four implicit curves enclose a nonempty open central configuration region B_2 (shown in Figure 7). Given any $1 \leq u$, the three equations $pm_1 = 0, pm_2 = 0, pm_3 = 0$ give rise to three implicit curves and the three implicit curves enclose a nonempty open central configuration region B_3 (shown in Figure 8).

Proof. First, we show that $pm_2 = 0, pm_3 = 0$ give rise two smooth monotone increasing curves enclosing an open region in which $pm_2 > 0, pm_3 > 0$ with the curve $pm_3 = 0$ above the curve $pm_2 = 0$. In fact, let

$$c_0(s,u) = 16{+}16\,u{+}48\,s{+}us^4{+}56\,s^2{+}s^5{+}32\,s^3{+}24\,us^2{+}32\,us{+}8\,us^3{+}9\,s^4 = (s+2)^4\,(s+1+u)\,,$$

$$c_1(s,u) = 16 + 40\,s + 2\,us^3 + 24\,us + 36\,s^2 + 2\,s^4 + 14\,s^3 + 12\,us^2 + 16\,u = 2\,(s+2)^3\,(s+1+u)\,,$$

$$c_2(s,u) = \left(us^2 + 4 + 4\,us + 5\,s^2 + 4\,u + 8\,s + s^3\right) = (s+2)^2\,(s+1+u)\,,$$

$$c_3(s,u) = \left(-s^2 - 6\,s + 2\,us + 4\,u - 8\right) = -\,(s+2)\,(s - 2\,u + 4)\,,$$

$$c_4(s,u) = (-5 + u - 2\,s)\,.$$

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then $$pm_2(t,s,u) = c_0(s,u) + c_1(s,u)t + c_2(s,u)t^2 + c_3(s,u)t^3 + c_4(s,u)t^4 - t^5.$$

The zeros of the coefficients are $s = -u-1$, $s = -u-1$, $s = -u-1$, $s = 2u-4$, $s = -5/2+1/2u$ respectively. In $us-$plane, these linear functions of u and s divide the half plane $(s > 0)$ into 4 parts, $A_1, \cdots, A_4$ as indicated in figure 9.

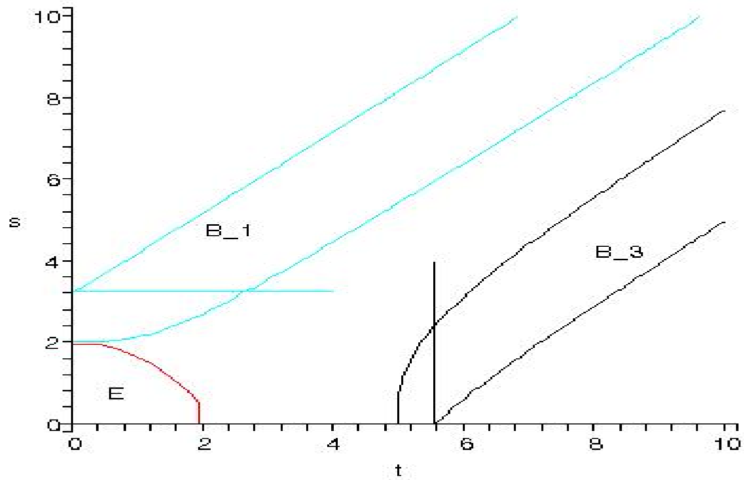

Figure 8: C.C. Region for fixed $1 \leq u$ and $u \leq -1$

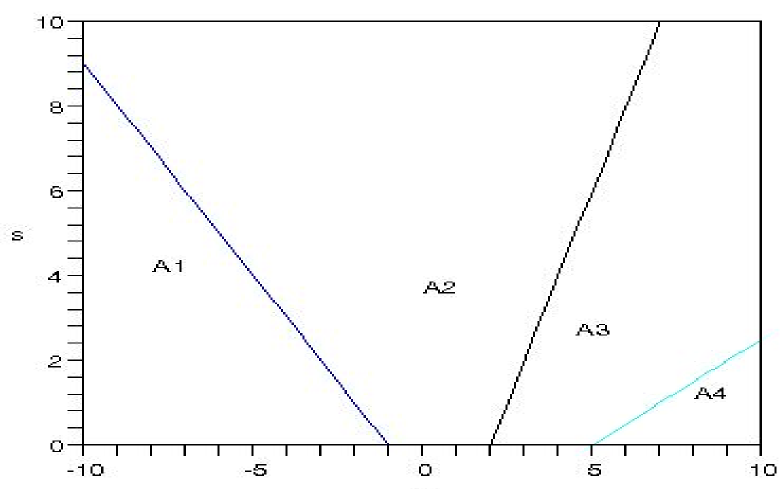

Figure 9: Change of the Sign

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For given u, s in any of $A_1, \cdots, A_4$, the signs of the coefficients change at most once. More precisely, all the coefficients have negative signs in region A_1 and the coefficients only change sign once in the other regions. By Descartes' rule, $pm_2(t, s, u) = 0$ has at most one positive zero for given $s > 0$,and any u. Then $pm_2(t, s, u) = 0$ gives rise to an implicit surface (the left surface in figure 10) with $s > 0, t > 0$. Therefore $pm_2(t, s, u) = 0$ gives rise to the implicit curve in figure 7 for given u (the curve $pm_2 = 0$ in figure 7 is for $u = 3/4$). Using similar arguments for $pm_3 = 0$ we can conclude that given u, $pm_2 = 0$ and $pm_3 = 0$ define two implicit curves in first quadrant of ts-plane. Now we want to show the curve $pm_3 = 0$ is always above $pm_2 = 0$ for any given u (or equivalently show that the surface $pm_3 = 0$ is above the surface pm_2).

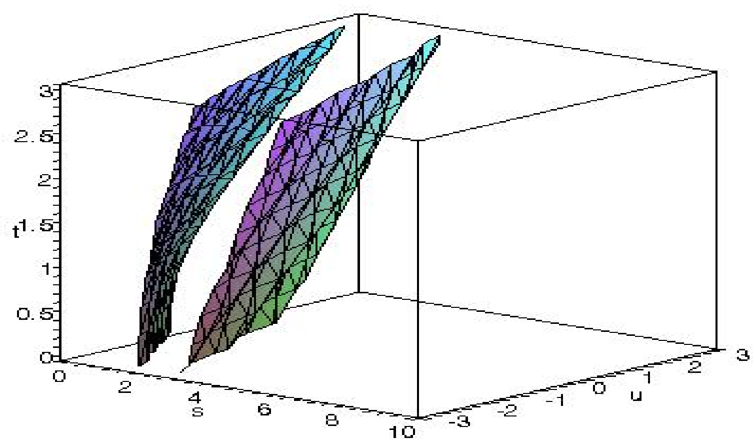

Figure 10: The Surface of $m_2 = 0$ and $m_3 = 0$

From the proof of Lemma 2.1, for given $u = 0$, the curve $pm_3 = 0$ is above the curve $pm_2 = 0$. The surface $pm_3 = 0$ is above the surface $pm_2 = 0$ if the surface $pm_3 = 0$ doesn't intersect with the surface $pm_2 = 0$ i.e., if the equation system

$$\begin{cases} pm_2(t, s, u) = 0 \\ pm_3(t, s, u) = 0 \\ pm_2(t, s, u) = pm_3(t, s, u) \end{cases} \tag{2.15}$$

has no solution for $t > 0, s > 0$. In fact, from $pm_3 - pm_2 = 0$ we have $u = \frac{-16t + \cdots + s^3 t^2}{24t + \cdots + 8st^2}$. Substituting u into pm_2, we have $pm_2 = \frac{256 + 896s + \cdots + 9s^2 t^6}{16 + 24s + \cdots + t^4}$ in which all terms are positive. So it is

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impossible that $pm_2 = 0$ and $pm_3 = 0$ simultaneously.

For any given $u \leq -1$, $pm_1 > 0$ for $t > 0, s > 0$ by Lemma 1. Also $pm_4 = 0$ gives rise to a horizontal line $s = s_0$ with $s_0 \geq 2$ by Lemma 2.1. From Lemma 2.2, $pm_3 = 0$ intersects $pm_4 = 0$ at $t = 0$. In addition, $pm_3 = 0$ is above $pm_2 = 0$. So the three implicit curves $pm_2 = 0, pm_3 = 0, pm_4 = 0$ enclose an open unbounded strip-like central configuration region B_1 as indicated in figure 8. The region slides from infinity to $s = 2$ with one vertex on the s-axis while u changes from negative infinity to -1.

For any given $-1 < u < 1$, $pm_1 = 0, pm_4 = 0$ give rise to two implicit straight line $t = t_0, s = s_0$ respectively with $(t_0, s_0) \in \Gamma$ from Lemma 1. Then the four implicit curves $pm_1 = 0, pm_2 = 0, pm_3 = 0, pm_4 = 0$ enclose an open unbounded strip-like central configuration region B_2 as indicated in figure 7. The region slides with one vertex on Γ.

For any given $u \geq 1$, $pm_4 > 0$ for $t > 0, s > 0$ by lemma 2.1. Also $pm_1 = 0$ gives rise to a vertical line $t = t_0$ with $t_0 \geq 2$ by Lemma 2.1. From Lemma 2, $pm_2 = 0$ intersects $pm_1 = 0$ at $s = 0$. In addition $pm_3 = 0$ is above $pm_2 = 0$. So the three implicit curves $pm_1 = 0, pm_2 = 0, pm_3 = 0$ enclose an open unbounded strip-like central configuration region B_3 as indicated in figure 8. The region slides from $t = 2$ to infinity with one vertex on the t-axis while u changes from 1 to infinity.

Remark 4: From Lemma 2.2 and Lemma 2.3, for given $0 < u < 1$, the unique intersecting point between $pm_1 = 0$ and $pm_3 = 0$ is above the unique intersecting point between $pm_1 = 0$ and $pm_4 = 0$ because $pm_3 = 0$ intersects $pm_4 = 0$ at $t = 0$ and $pm_3 = 0$ is monotonically increasing (for example: P is above O in Figure 7 with $u = 3/4$). Similarly, the unique intersecting point between $pm_4 = 0$ and $pm_2 = 0$ is to the right of the unique intersecting point between $pm_4 = 0$ and $pm_1 = 0$ (for example: Q is at the right of O in Figure 7 with $u = 3/4$). It follows that as u changes continuously, the central configuration region is swept out continuously.

Lemma 2.4. For any point (t_0, s_0) in region G in figure 6, there exists at least one u such that the corresponding configuration $(-s_0 - 1, -1, 1, t_0 + 1)$ could become a central configuration centered at u. Therefore, region G is a central configuration region.

Proof. Lemma 2.4 can be obtained from the proof of Lemma 3 because the central configuration region B obtained in lemma 3 sweeps all the region G. Once (t_0, s_0) falls in a central configuration region B which is obtained for a fixed u, then the configuration $(-s_0 - 1, -1, 1, t_0 + 1)$ is a central configuration by choosing proper positive masses centered at u. $\sharp$

Remark 5: The four lemmas complete the proof of theorem 2.2. For any given point (t_0, s_0)

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in region G, the configuration $x = (-s_0 - 1, -1, 1, t_0 + 1)$ could be a central configuration for some center of mass u.

2.2.4 Proof of Theorem 2.3

For any given configuration $x = (-s - 1, -1, 1, t + 1)$ and center of mass u, we get the unique mass solution $m = m(x, u)$ in section 2. Given any x, if there exist u such that $\underline{u}(x) < u < \bar{u}(x)$, then the configuration x is a central configuration with mass $m(x, u)$ which is positive and centered at u. Now we turn to prove the theorem 2.3. From Lemma 2.1 in section 2.2.3 we know that there does not exist a positive mass making configuration $x = (-s_0 - 1, -1, 1, t_0 + 1)$ central at any point $(t_0, s_0) \in E$. Then the set $\{u|x$ forms a central configuration centered at u with positive mass $m(x, u)\}$ is empty implying $d(t_0, s_0) = 0$.

From Lemma 2.4 in section 2.2.3, for each point (t_0, s_0) in the region G, there exist at least one u_0 such that $(t_0, s_0) \in B$, which is an open central configuration region corresponding to u_0. Because B is open, (t_0, s_0) is an interior point of B. Then for u in a small neighborhood of u_0, (t_0, s_0) is still in the central configuration region for those u. Note that the central configuration changes continuously w.r.t. the change of center of mass u. So the small neighborhood is in the set $\{u|x$ forms a central configuration centered at u with positive mass m(x,u)$\}$. So $d(t_0, s_0) > 0$.

In order to show $\lim_{t_0 \to \infty, s_0 \to 0} d(t_0, s_0) = 0$, we need to show the central configuration region moves with the almost the same speed as u moves. We also need to show that the slope of $pm_2 = 0$ goes to infinity as t_0 goes to infinity and as s_0 goes to zero. In fact, from $pm_1 = 0$, it is easy to solve for u, which is

$$u = \frac{-4\,t^2 - 16\,t - 16 + 8\,t^3 + t^5 + 5\,t^4}{16\,t + 16 + t^4 + 4\,t^3 + 4\,t^2}$$

$$\frac{du}{dt} = \frac{t^2\left(768\,t + 112\,t^3 + 576 + 416\,t^2 + t^6 + 8\,t^5 + 24\,t^4\right)}{\left(16\,t + 16 + t^4 + 4\,t^3 + 4\,t^2\right)^2}.$$

As t goes to infinity, $\frac{du}{dt}$ goes to 1.

The implicit derivative $\frac{ds}{dt}$ in $pm_2 = 0$ is

$$\frac{ds}{dt} = -\frac{16 + \cdots - 5\,t^4 + 4\,t^3u - 18\,st^2 + 10\,s^2t + 2\,s^4 + 36\,s^2 - 3\,s^2t^2}{48 + \cdots + 48\,us + 4\,us^3 + 8\,s^3t - 2\,st^3 + 3\,s^2t^2}$$

In order to consider the change of the slope of the curve $pm_2 = 0$ as t goes to infinity and as s goes to zero, we substitute u obtained above from $pm_1 = 0$ into $\frac{ds}{dt}$ and let s go to zero which gives us

$$\frac{ds}{dt}|_{s \to 0} = 1/4\,\frac{\left(t^5 + 6\,t^4 + 12\,t^3 + 88\,t^2 + 240\,t + 288\right)t^2}{9\,t^5 + 30\,t^4 + 44\,t^3 + 24\,t^2 + 48\,t + 32} \approx \frac{t^2}{36} \text{ for large t.}$$

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So for small s_0 and large t_0, $d(t_0, s_0) \approx \frac{36 s_0}{t_0^2}$. Therefore $\lim_{t_0 \to \infty, s_0 \to 0} d(t_0, s_0) = 0$.

3 Regularization of Simultaneous Binary Collision

We consider the classical collinear four-body problem of celestial mechanics. Let $x_k \in \mathbb{R}$, $k = 1, 2, 3, 4$, denote the position of k^{th} body on the line with mass $m_k > 0$. Assume, without loss of generality, that $x_1 \leq x_2 \leq x_3 \leq x_4$, then the Newtonian system (1.1) for collinear four bodies is

$$m_k \frac{d^2 x_k}{dt^2} = \frac{\partial U}{\partial x_k}, \qquad k = 1, 2, 3, 4, \tag{3.1}$$

and the Newtonian Potential U in (1.2) is

$$U = \sum_{1 \leq j < i \leq 4} \frac{m_i m_j}{|x_i - x_j|}. \tag{3.2}$$

The total energy

$$H = \sum_{1 \leq i \leq 4} \frac{1}{2} m_i |\dot{x}_i|^2 - \sum_{1 \leq j < i \leq 4} \frac{m_i m_j}{|x_i - x_j|} \tag{3.3}$$

is constant along a solution of (3.1).

We call the space of $x = (x_1, \cdots, x_4) \in \mathbb{R}^4$ the space of positions. Let $\triangle_{ij} := \{x \in \mathbb{R}^4, x_i = x_j\}$ and $\triangle := \bigcup_{1 \leq j < i \leq 4} \triangle_{ij}$. The potential function U, and consequently equation (3.1) are singular on $\triangle$.

Let $x(t) = (x_1(t), \cdots, x_4(t))$ be a solution of equation (3.1) defined on $[t_1, t_2)$, and assume that $x(t) \to L = (L_1, \cdots, L_4)$ as $t \to t_2^-$. We say that $x(t)$ has a singularity of collision at $t = t_2$ if $L \in \triangle$. According to the locations of L in $\triangle$, the singularities of collision are divided into the categories of (I) binary collisions, (II) simultaneous binary collisions, (III) triple collisions and (IV) four-body (total) collisions [35]. In this section, we study a solution with singularity of *simultaneous binary collision(SBC),* that is, the limit L of the position satisfies $-\infty < L_1 = L_2 < L_3 = L_4 < \infty$. Let us denote the set of L satisfying these restrictions as $\bigwedge$.

For better understanding of the behavior of the motion of the particles in a neighborhood of a collision, we make a change of coordinates and of time scale. If, in the new coordinates, the orbits which approach collision can be extended across the collision in a smooth manner with respect to the new time scale, we say that the collision orbits have been regularized. The regularization is of class $C^n, n \geq 0$, or analytic if each collision orbit of the transformed differential equations is C^n

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or analytic, respectively, as a function of the new time scale in a neighborhood of collision. This type of regularization goes back, in particular, to Sundman [43] in his studies of collisions in the three-body problem (see also [37]).

It is worth to mention that the above regularization is just to extend each individual collision orbit itself across collision. A related question is about the smoothness of the flow with respect to initial conditions in a neighborhood of a collision orbit. This defines a different type of regularization if the flow also varies smoothly with respect to initial conditions in a neighborhood of a collision orbit. This type of regularization was first studied by Easton [10], and later by many other people, see Regina Martinez and Carles Simó [21], R. McGehee [36] etc. Many other works can be found from the reference of these papers.

In this section we construct coordinate transforms in new time scale that remove the singularities of simultaneous binary collisions in collinear four-body problem without any assumption on mass. The regularization is at least of class C^2. Based on the results of regularization of SBC in this section, the behavior of the motion is studied for the motion acrossing collisions. The existence of a family of periodic solutions with simultaneous binary collision is proved in subsection 3.2. More periodic solutions involving single binary collision and SBC are constructed in section 3.3.

We now proceed to the next subsection and state our results more precisely.

3.1 Regularization

Let us now consider equation (3.1) for the collinear four-body problem assuming $x_1 \leq x_2 \leq x_3 \leq x_4$. Without loss of generality, we also put the center of mass at the origin which implies

$$\sum_{k=1}^{4} m_k x_k = 0. \tag{3.4}$$

Related to (3.4), we have

$$\sum_{k=1}^{4} m_k \frac{dx_k}{dt} = 0. \tag{3.5}$$

These help in cutting the dimension of the phase space down by two. Let

$$u_1 = x_2 - x_1, \quad u_2 = x_4 - x_3, \quad u_3 = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \tag{3.6}$$

and

$$K_{31} = \frac{1}{x_3 - x_1} = \frac{\omega_1}{\omega_2 u_1 - \omega_3 u_2 - \omega_4 u_3} \tag{3.7}$$

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$$K_{41} = \frac{1}{x_4 - x_1} = \frac{\omega_1}{\omega_2 u_1 + \omega_5 u_2 - \omega_4 u_3} \tag{3.8}$$

$$K_{32} = \frac{1}{x_3 - x_2} = \frac{\omega_1}{-\omega_6 u_1 - \omega_3 u_2 - \omega_4 u_3} \tag{3.9}$$

$$K_{42} = \frac{1}{x_4 - x_2} = \frac{\omega_1}{-\omega_6 u_1 + \omega_5 u_2 - \omega_4 u_3} \tag{3.10}$$

where $\omega_1 = (m_1 + m_2)(m_3 + m_4), \omega_2 = m_2(m_3 + m_4), \omega_3 = m_4(m_1 + m_2), \omega_4 = (m_1 + m_2)(m_1 + m_2 + m_3 + m_4)$, $\omega_5 = m_3(m_1 + m_2)$ and $\omega_6 = m_1(m_3 + m_4)$.

Then equation (3.1) reduces to an ordinary differential equation system with six independent variables $\vec{p}_1 = (u_1, u_2, u_3, v_1, v_2, v_3)$,

$$\begin{aligned}
\frac{du_1}{dt} &= v_1, & \frac{dv_1}{dt} &= -\frac{m_1 + m_2}{u_1^2} + m_3(K_{32}^2 - K_{31}^2) + m_4(K_{42}^2 - K_{41}^2); \\
\frac{du_2}{dt} &= v_2, & \frac{dv_2}{dt} &= -\frac{m_3 + m_4}{u_2^2} + m_1(K_{31}^2 - K_{41}^2) + m_2(K_{32}^2 - K_{42}^2); \\
\frac{du_3}{dt} &= v_3, & \frac{dv_3}{dt} &= \frac{m_1 m_3}{m_1 + m_2} K_{31}^2 + \frac{m_2 m_3}{m_1 + m_2} K_{32}^2 + \frac{m_1 m_4}{m_1 + m_2} K_{41}^2 + \frac{m_2 m_4}{m_1 + m_2} K_{42}^2.
\end{aligned} \tag{3.11}$$

$\vec{u} = (u_1, u_2, u_3) \in \mathbb{R}^2 \times \mathbb{R}^-$ is now the space of positions and $\bigwedge = \{u_1 = u_2 = 0, u_3 \in \mathbb{R}^-\}$ is the singular set for simultaneous binary collisions. $K_{ij}, i = 3, 4, j = 1, 2$ are bounded on the singular set $\bigwedge$.

It is verified that the total energy (3.3) becomes,

$$\hat{H} = \frac{(\beta_1 v_1^2 + \beta_2 v_2^2 + \beta_3 v_3^2)}{2(m_1 + m_2)(m_3 + m4)} - (\frac{m_1 m_2}{u_1} + \frac{m_3 m_4}{u_2} + m_1 m_3 K_{31} + m_1 m_4 K_{41} + m_2 m_3 K_{32} + m_2 m_4 K_{42}) \tag{3.12}$$

where $\beta_1 = m_1 m_2(m_3 + m_4), \beta_2 = m_3 m_4(m_1 + m_2), \beta_3 = (m_1 + m_2)^2(m_1 + m_2 + m_3 + m_4)$.

One of the main results of this section reads as follows.

Theorem 3.1. *Any simultaneous binary collision orbit of collinear four body problem can be extended at least C^1 across $\bigwedge$ with respect to the new time scale after a change of coordinates and time scale.*

Furthermore, we also prove our extension is at least C^2 in theorem . The following lemma 3.1 and its proof are from the work of Belbruno [2].

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Lemma 3.1. *Let $\vec{u} = \vec{u}(t)$, $t \in [t_1, t_2)$ denote a simultaneous binary collision orbit encountering $\bigwedge$, where $t = t_2$ corresponds to collision, then*

$$\lim_{t \to t_2} \frac{du_k}{dt} = \lim_{t \to t_2} v_k(t) = \infty, \qquad k = 1, 2 \tag{3.13}$$

$$\lim_{t \to t_2} u_1(t) v_1^2(t) = 2(m_1 + m_2), \quad \lim_{t \to t_2} u_2(t) v_2^2(t) = 2(m_3 + m_4), \tag{3.14}$$

$$\lim_{t \to t_2} u_1(t) v_1(t) = 0, \quad \lim_{t \to t_2} u_2(t) v_2(t) = 0, \tag{3.15}$$

and

$$\lim_{t \to t_2} \frac{u_1(t)}{u_2(t)} = \alpha, \qquad \lim_{t \to t_2} \frac{v_1(t)}{v_2(t)} = \alpha, \tag{3.16}$$

where $\alpha = \left(\frac{m_1+m_2}{m_3+m_4}\right)^{\frac{1}{3}}$.

Proof. System (3.11) implies

$$\frac{d^2 u_1}{dt^2} = -\frac{m_1 + m_2}{u_1^2} + G_1, \qquad \frac{d^2 u_2}{dt^2} = -\frac{m_3 + m_4}{u_2^2} + G_2, \tag{3.17}$$

where $G_1 = m_3(K_{32}^2 - K_{31}^2) + m_4(K_{42}^2 - K_{41}^2)$ and $G_2 = m_1(K_{31}^2 - K_{41}^2) + m_2(K_{32}^2 - K_{42}^2)$ are bounded when t is close to t_2. Multiplying (3.17) by $\frac{du_1}{dt}$, $\frac{du_2}{dt}$ respectively, yields

$$\left(\frac{du_1}{dt}\right)^2 = \frac{2(m_1 + m_2)}{u_1} + \tilde{G}_1, \qquad \left(\frac{du_2}{dt}\right)^2 = \frac{2(m_3 + m_4)}{u_2} + \tilde{G}_2$$

where $\tilde{G}_1$ and $\tilde{G}_2$ are also bounded when t is close to t_2. Thus letting $u_k(t) \to 0$ as $t \to t_2, k = 1, 2$, we prove that $\lim_{t \to t_2} \frac{du_1}{dt} = \lim_{t \to t_2} v_1(t) = \infty$. and $\lim_{t \to t_2} \frac{du_2}{dt} = \lim_{t \to t_2} v_2(t) = \infty$. Multiplying by u_1, u_2 respectively, the above equations become

$$u_1 v_1^2 = 2(m_1 + m_2) + u_1 \tilde{G}_1, \quad u_2 v_2^2 = 2(m_3 + m_4) + u_2 \tilde{G}_2.$$

Then we have

$$\lim_{t \to t_2} u_1(t) v_1^2(t) = 2(m_1 + m_2), \quad \lim_{t \to t_2} u_2(t) v_2^2(t) = 2(m_3 + m_4).$$

Consequently,

$$\lim_{t \to t_2} u_1(t) v_1(t) = 0, \quad \lim_{t \to t_2} u_2(t) v_2(t) = 0.$$

By making use of the fact that both u_1, u_2 tend to 0 monotonically and the results above, we have $\lim_{t \to t_2} \frac{u_1(t)}{u_2(t)} = \alpha$, where $\alpha = \left(\frac{m_1+m_2}{m_3+m_4}\right)^{1/3}$. We also have

$$\lim_{t \to t_2} \frac{v_1(t)}{v_2(t)} = \lim_{t \to t_2} \frac{\dot{u}_1(t)}{\dot{u}_2(t)} = \lim_{t \to t_2} \frac{u_1(t)}{u_2(t)} = \alpha. \qquad \sharp$$

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3.1.1 The Proof of Theorem 3.1

Let $\vec{u} = \vec{u}(t)$ denote a simultaneous binary collision orbit encountering $\bigwedge$ when $t = t_2$, then in a sufficiently small open deleted neighborhood of $t = t_2$, $\vec{u}(t)$ performs no collisions [2]. Therefore we can assume $\vec{p}_1 = (u_1, \cdots, v_3)$ is a solution of (3.11) performing no collisions for $t \in [t_1, t_2)$ and performing a simultaneous binary collision when $t \to t_2^-$. We will construct coordinate transform with a new time scale τ, such that the orbit under the new coordinate can be regularized. Let $\delta > 1$ and $0 < \rho < 1$ be fixed. We only consider solutions of equations (3.11) in $\mathcal{U}_{\delta,\rho}$, where

$$\mathcal{U}_{\delta,\rho} = \{\vec{p}_1 \in (\mathbb{R})^{2+} \times \mathbb{R}^- \times \mathbb{R}^3 : u_1, u_2 < \rho; -\delta < u_3 < -\delta^{-1}\}$$

We are now ready to introduce regularization variables by a Levi-Civita transformation and a time scale. Let $\vec{p}_2 = (\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3)$ be the new phase variables with the new time variable as τ.

$$u_1 = \frac{\xi_1^2}{2}, u_2 = \frac{\xi_2^2}{2}, u_3 = -\frac{\xi_3^2}{2}, v_1 = \frac{\eta_1}{\xi_1}, v_2 = \frac{\eta_2}{\xi_2}, v_3 = \frac{\eta_3}{\xi_3} \tag{3.18}$$

and rescale time by

$$dt = (\xi_1^2 + \xi_2^2) d\tau \tag{3.19}$$

One verifies that (3.11) becomes

$$\frac{d\xi_1}{d\tau} = \frac{\xi_1^2 + \xi_2^2}{\xi_1^2}\eta_1, \tag{3.20}$$

$$\frac{d\xi_2}{d\tau} = \frac{\xi_1^2 + \xi_2^2}{\xi_2^2}\eta_2 \tag{3.21}$$

$$\frac{d\xi_3}{d\tau} = -\frac{(\xi_1^2 + \xi_2^2)}{\xi_3^2}\eta_3 \tag{3.22}$$

$$\frac{d\eta_1}{d\tau} = \frac{(\eta_1^2 - 4(m_1 + m_2))}{\xi_1^2}\frac{\xi_1(\xi_1^2 + \xi_2^2)}{\xi_1^2} + \left(m_3(K_{32}^2 - K_{31}^2) + m_4(K_{42}^2 - K_{41}^2)\right)\xi_1(\xi_1^2 + \xi_2^2) \tag{3.23}$$

$$\frac{d\eta_2}{d\tau} = \frac{(\eta_2^2 - 4(m_3 + m_4))}{\xi_2^2}\frac{\xi_2(\xi_1^2 + \xi_2^2)}{\xi_2^2} + \left(m_1(K_{31}^2 - K_{41}^2) + m_2(K_{32}^2 - K_{42}^2)\right)\xi_2(\xi_1^2 + \xi_2^2) \tag{3.24}$$

$$\frac{d\eta_3}{d\tau} = \left(\frac{m_1 m_3}{m_1 + m_2}K_{31}^2 + \frac{m_2 m_3}{m_1 + m_2}K_{32}^2 + \frac{m_1 m_4}{m_1 + m_2}K_{41}^2 + \frac{m_2 m_4}{m_1 + m_2}K_{42}^2\right)\xi_3(\xi_1^2 + \xi_2^2) + \frac{(\xi_1^2 + \xi_2^2)}{\xi_3^3}\eta_3^2 \tag{3.25}$$

where $K_{31}, K_{32}, K_{41}, K_{42}$ are obtained by substituting (3.18) into (3.7)-(3.10), which are bounded and smooth on $\bigwedge$.

Derivations for equations (3.20) to (3.25): For the first equation (3.20) we differentiate $\xi_1^2 = 2u_1$ to obtain

$$\frac{d\xi_1}{d\tau} = \frac{1}{\xi_1}\frac{du_1}{dt}\frac{dt}{d\tau} = \frac{1}{\xi_1}\frac{\eta_1}{\xi_1}(\xi_1^2 + \xi_2^2) = \frac{\xi_1^2 + \xi_2^2}{\xi_1^2}\eta_1.$$

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$$\frac{d\eta_1}{d\tau} = \frac{d}{d\tau}(\xi_1 v_1) = \frac{dv_1}{d\tau}\xi_1 + v_1\frac{d\xi_1}{d\tau}$$
$$= \left(-\frac{4(m_1+m_2)}{\xi_1^4} + m_3(K_{32}^2 - K_{31}^2) + m_4(K_{42}^2 - K_{41}^2)\right)\xi_1(\xi_1^2+\xi_2^2) + \frac{\eta_1^2(\xi_1^2+\xi_2^2)}{\xi_1^3}$$
$$= \frac{(\eta_1^2 - 4(m_1+m_2))}{\xi_1^2}\frac{\xi_1(\xi_1^2+\xi_2^2)}{\xi_1^2} + \left(m_3(K_{32}^2 - K_{31}^2) + m_4(K_{42}^2 - K_{41}^2)\right)\xi_1(\xi_1^2+\xi_2^2)$$

$$\frac{d\eta_3}{d\tau} = \xi_3\frac{dv_3}{dt}\frac{dt}{d\tau} + v_3\frac{d\xi_3}{d\tau}$$
$$= \left(\frac{m_1m_3}{m_1+m_2}K_{31}^2 + \frac{m_2m_3}{m_1+m_2}K_{32}^2 + \frac{m_1m_4}{m_1+m_2}K_{41}^2 + \frac{m_2m_4}{m_1+m_2}K_{42}^2\right)\xi_3(\xi_1^2+\xi_2^2) + \frac{(\xi_1^2+\xi_2^2)}{\xi_3^3}\eta_3^2$$

Other equations can be obtained in a similar way.

The energy (3.12) becomes

$$\begin{aligned}\hat{H} &= \tfrac{(\eta_1^2-4(m_1+m_2))}{\xi_1^2}\tfrac{m_1m_2}{2(m_1+m_2)} + \tfrac{\eta_2^2-4(m_3+m_4))}{\xi_2^2}\tfrac{m_3m_4}{2(m_3+m_4)} \\ &+\tfrac{\eta_3^2}{\xi_3^2}\tfrac{\beta_3}{2(m_1+m_2)(m_3+m4)} - (m_1m_3K_{31} + m_1m_4K_{41} + m_2m_3K_{32} + m_2m_4K_{42})\end{aligned} \tag{3.26}$$

where $\beta_3 = (m_1+m_2)^2(m_1+m_2+m_3+m_4)$.

Remark: We choose $\xi_3 = -\sqrt{-2u_3}$ the negative branch of equation (3.18). The set $\{\xi_1 = \xi_2 = 0, \xi_3 < 0\}$ is the singular set corresponding to $\bigwedge$ the singular set for the simultaneous binary collisions. $K_{31}, K_{32}, K_{41}, K_{42}$ are bounded smooth functions on the singular set.

Let

$$\mathcal{V}_{\delta,\rho} = \{\vec{p}_2 = (\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3) : \xi_1^2, \xi_2^2 < 2\rho, -(2\delta)^{1/2} < \xi_3 < -(2\delta^{-1})^{1/2}\}$$

be the correspondence of $\mathcal{U}_{\delta,\rho}$ in phase space $\vec{p}_1$. We will study the solutions of (3.20)-(3.25) in $\mathcal{V}_{\delta,\rho}$.

Recall that $\vec{p}_1 = (u_1, u_2, u_3, v_1, v_2, v_3)$ is a solution in $\mathcal{U}_{\delta,\rho}$ defined in $[t_1, t_2)$ and assume that $\vec{p}_1 \to \bigwedge$ as $t \to t_2^-$. Let

$$\tau(t) = \tau_0 + \int_{t_1}^{t} \frac{1}{2(u_1(s)+u_2(s))}ds \tag{3.27}$$

Theorem 3.2. *Let $\vec{p}_1(t)$, $t \in [t_1, t_2)$ be a solution of equation (3.11) in $\mathcal{U}_{\delta,\rho}$ with simultaneous binary collision at $t = t_2$, in other words, $\vec{p}_1(t) \to \bigwedge$ as $t \to t_2^-$. Let $\tau(t)$ be defined by (3.27) and $\vec{p}_2(\tau)$, $\tau \in [\tau_1, \tau_2)$ be the functions obtained from $\vec{p}_1(t)$ through (3.18). Then*

(1) $\vec{p}_2(\tau)$ is a solution of equations (3.20)-(3.25) in $[\tau_1, \tau_2)$;

(2) $\tau_2 := \tau(t_2) < \infty$, and $\vec{p}_2(\tau_2) := \lim_{\tau\to\tau_2}\vec{p}_2(\tau)$ is well defined; and

(3) the solution $\vec{p}_2$ of (3.20)-(3.25) can be at least C^1 smoothly extended through τ_2.

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Proof. (1) This follows from the derivations of equations. We notice that (3.18) allows different ways to convert $\vec{p}_1(t)$ to $\vec{p}_2(\tau)$ because $\xi_k, k = 1,2,3$ can have different signs. This is a well known characteristic of Levi-Civita variables. For definiteness, let us choose the positive sign $\xi_k = \sqrt{2u_k}, k = 1,2$ and negative sign $\xi_3 = -\sqrt{-2u_3}$.

(2) It is well know that when a collision singularity occurs at t_2,

$$u_1(t) + u_2(t) \sim (t - t_2)^{2/3}.$$

Then it follows that

$$\tau_2 = \tau_0 + \int_{t_1}^{t_2} \frac{1}{2(u_1(t) + u_2(t))} dt < \infty.$$

It is easy to show that $v_3(t)$ and $u_k(t) \to$ a definite limit as $t \to t_2^-$, which we denote by $v_3(t_2), u_k(t_2), k = 1,2,3$. Now for $\vec{p}_2(\tau_2)$: we let $\xi_k(\tau_2) = \sqrt{2u_k(t_2)}, k = 1,2$ and $\xi_3(\tau_2) = -\sqrt{-2u_3(t_2)}, \eta_3(\tau_2) = \xi_3 v_3(t_2)$. By the assumption, $\xi_1(\tau_2) = 0, \xi_2(\tau_2) = 0$.

From above, we have

$$\lim_{\tau \to \tau_2} \eta_1^2(\tau) = \lim_{t \to t_2} 2u_1 v_1^2 = 4(m_1 + m_2),$$

and

$$\lim_{\tau \to \tau_2} \eta_2^2(\tau) = \lim_{t \to t_2} 2u_2 v_2^2 = 4(m_3 + m_4),$$

from which it follows that $\eta_1(\tau_2) = -2\sqrt{m_1 + m_2}$, $\eta_2(\tau_2) = -2\sqrt{m_3 + m_4}$. They are negative because we have chosen positive sigh for $\xi_k(\tau_2), k = 1,2$. Therefore $\vec{p}_2(\tau_2) := \lim_{\tau \to \tau_2} \vec{p}_2(\tau)$ is well defined.

Before we prove (3), we need the following lemma 3.2.

Lemma 3.2. *Let $\vec{p}_1(t)$ be a solution of (3.11). $\vec{p}_2(\tau)$ is obtained from $\vec{p}_1(t)$ as above. Then*

$$\lim_{\tau \to \tau_2} \frac{\xi_1^2(\tau)}{\xi_2^2(\tau)} = \alpha \tag{3.28}$$

$$\lim_{\tau \to \tau_2} \frac{\xi_1^2(\tau) + \xi_2^2(\tau)}{\xi_2^2(\tau)} = 1 + \alpha \tag{3.29}$$

$$\lim_{\tau \to \tau_2} \frac{\xi_1^2(\tau) + \xi_2^2(\tau)}{\xi_1^2(\tau)} = 1 + \frac{1}{\alpha} \tag{3.30}$$

$$\lim_{\tau \to \tau_2} \frac{\frac{\eta_1^2(\tau)}{m_1+m_2}}{\frac{\eta_2^2(\tau)}{m_3+m_4}} = 1 \tag{3.31}$$

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Proof. By directional computation and lemma 3.1, we can check that

$$\lim_{\tau\to\tau_2}\frac{\xi_1^2(\tau)}{\xi_2^2(\tau)}=\lim_{t\to t_2}\frac{u_1}{u_2}=\alpha,$$

$$\lim_{\tau\to\tau_2}\frac{\xi_1^2(\tau)+\xi_2^2(\tau)}{\xi_2^2(\tau)}=\lim_{\tau\to\tau_2}(1+\frac{\xi_1^2(\tau)}{\xi_2^2(\tau)})=1+\alpha$$

$$\lim_{\tau\to\tau_2}\frac{\xi_1^2(\tau)+\xi_2^2(\tau)}{\xi_1^2(\tau)}=\lim_{\tau\to\tau_2}(1+\frac{\xi_2^2(\tau)}{\xi_1^2(\tau)})=1+\frac{1}{\alpha}$$

$$\lim_{\tau\to\tau_2}\frac{\frac{\eta_1^2(\tau)}{m_1+m_2}}{\frac{\eta_2^2(\tau)}{m_3+m_4}}=\lim_{t\to t_2}\frac{2u_1v_1^2(t)(m_3+m_4)}{2u_2v_2^2(t)(m_1+m_2)}=1. \qquad \sharp$$

The proof of (3). From (3.12), we have

$$\begin{aligned}\frac{(\eta_1^2-4(m_1+m_2))}{\xi_1^2}\frac{m_1m_2}{2(m_1+m_2)}+\frac{\eta_2^2-4(m_3+m_4))}{\xi_2^2}\frac{m_3m_4}{2(m_3+m_4)}=\hat{H}-\Big(\frac{(\beta_3\eta_3^2)}{2\xi_3^2(m_1+m_2)(m_3+m4)}\\ -(m_1m_3K_{31}+m_1m_4K_{41}+m_2m_3K_{32}+m_2m_4K_{42})\Big)\,.\end{aligned} \tag{3.32}$$

Because $\hat{H}$ is a constant along any solution $\vec{p}_2(\tau)$, the right side in (3.32) is bounded in $[\tau_1,\tau_2)$ and the limit of the right side in (3.32) is finite defined by L as $\tau\to\tau_2$, i.e.,

$$\lim_{\tau\to\tau_2}\frac{(\eta_1^2-4(m_1+m_2))}{\xi_1^2}\frac{m_1m_2}{2(m_1+m_2)}+\frac{(\eta_2^2-4(m_3+m_4))}{\xi_2^2}\frac{m_3m_4}{2(m_3+m_4)}=L.$$

In addition,

$$\lim_{\tau\to\tau_2}\frac{\xi_1^2(\tau)}{\xi_2^2(\tau)}=\alpha,\quad \lim_{\tau\to\tau_2}\xi_1(\tau)=0,\quad \lim_{\tau\to\tau_2}\xi_2(\tau)=0,\quad \lim_{\tau\to\tau_2}\frac{\frac{\eta_1^2(\tau)}{m_1+m_2}}{\frac{\eta_2^2(\tau)}{m_3+m_4}}=1,$$

and

$$\lim_{\tau\to\tau_2}\frac{\frac{(\eta_1^2-4(m_1+m_2))}{\xi_1^2}\frac{m_1m_2}{2(m_1+m_2)}}{\frac{\eta_2^2-4(m_3+m_4))}{\xi_2^2}\frac{m_3m_4}{2(m_3+m_4)}}=\frac{m_1m_2(m_3+m_4)}{\alpha m_3m_4(m_1+m_2)}\lim_{t\to t_2}\frac{u_1v_1^2-2(m_1+m_2)}{u_2v_2^2-2(m_3+m_4)}$$

$$=\frac{m_1m_2(m_3+m_4)}{m_3m_4(m_1+m_2)}\lim_{t\to t_2}\frac{v_1^2-\frac{2(m_1+m_2)}{u_1}}{v_2^2-\frac{2(m_3+m_4)}{u_2}}=\frac{m_1m_2(m_3+m_4)}{m_3m_4(m_1+m_2)}\lim_{t\to t_2}\frac{2v_1\frac{dv_1}{dt}+\frac{2(m_1+m_2)}{u_1^2}}{2v_2\frac{dv_2}{dt}+\frac{2(m_3+m_4)}{u_2^2}}$$

$$=\frac{m_1m_2(m_3+m_4)}{m_3m_4(m_1+m_2)}\lim_{t\to t_2}\frac{\frac{2(m_1+m_2)}{u_1^2}(-v_1+1)}{\frac{2(m_3+m_4)}{u_2^2}(-v_2+1)}=\frac{m_1m_2}{\alpha^2m_3m_4}\lim_{t\to t_2}\frac{(-v_1+1)}{(-v_2+1)}$$

$$=\frac{m_1m_2}{\alpha m_3m_4}$$

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Then $\frac{(\eta_1^2-4(m_1+m_2))}{\xi_1^2}$ and $\frac{(\eta_2^2-4(m_3+m_4))}{\xi_2^2}$ are well defined when $\tau\to\tau_2$ by making use of (3.32), and $\frac{(\eta_1^2-4(m_1+m_2))}{\xi_1^2}\frac{\xi_1(\xi_1^2+\xi_2^2)}{\xi_1^2}$ and $\frac{(\eta_2^2-4(m_3+m_4))}{\xi_2^2}\frac{\xi_1(\xi_1^2+\xi_2^2)}{\xi_2^2}$ in (3.23)-(3.25) go to zero as $\tau\to\tau_2$. In fact, we can prove this by direct computation as follows:

$$\lim_{\tau\to\tau_2}\frac{(\eta_1^2-4(m_1+m_2))}{\xi_1^2}\frac{\xi_1(\xi_1^2+\xi_2^2)}{\xi_1^2}=\lim_{\tau\to\tau_2}\frac{(\eta_1^2-4(m_1+m_2))}{\xi_1}\lim_{\tau\to\tau_2}\frac{(\xi_1^2+\xi_2^2)}{\xi_1^2}$$
$$=(1+\frac{1}{\alpha})\lim_{t\to t_2}\frac{u_1v_1^2-2(m_1+m_2)}{\sqrt{2u_1}}=(1+\frac{1}{\alpha})\lim_{t\to t_2}\frac{u_1\tilde{G}_1}{\sqrt{2u_1}}=0.$$

According to lemma 3.2, it is clear that the functions on the right-hand side of (3.20)-(3.25) have a well-defined finite limit as $\tau\to\tau_2$ along $\vec{p}_2(\tau)$ given in the above. Moreover, $(\xi_1(\tau),\xi_2(\tau),\xi_3(\tau))$ intersects the simultaneous binary collision set $\bigwedge=\{\xi_1=0,\xi_2=0,\xi_3<0\}$ transversally, because letting $\tau\to\tau_2$ in (3.20), (3.21) implies

$$\lim_{\tau\to\tau_2}\frac{d\xi_1}{d\tau}=\lim_{\tau\to\tau_2}\frac{\xi_1^2+\xi_2^2}{\xi_1^2}\eta_1=2(1+\frac{1}{\alpha})\sqrt{m_1+m_2}>0,$$
$$\lim_{\tau\to\tau_2}\frac{d\xi_2}{d\tau}=\lim_{\tau\to\tau_2}\frac{\xi_1^2+\xi_2^2}{\xi_2^2}\eta_2=2(1+\alpha)\sqrt{m_3+m_4}>0.$$

Thus, $(\xi_1(\tau),\xi_2(\tau),\xi_3(\tau))$ can be extended across $\bigwedge$. The solution $\vec{p}_2(\tau)=(\xi_1(\tau),\xi_2(\tau),\xi_3(\tau),$ $\eta_1(\tau),\eta_2(\tau),\eta_3(\tau))$ can be extended for $\tau>\tau_2$ by solving the differential equations (3.20)-(3.25) with initial condition $\vec{p}_(\tau)=\vec{p}_2(\tau_2)$ when $\tau=\tau_2$. The vector field given by (3.20)-(3.25) is clearly continuous at $\tau=\tau_2$ and therefore, the components $\xi_1(\tau),\xi_2(\tau),\xi_3(\tau),\eta_1(\tau),\eta_2(\tau),\eta_3(\tau)$ are continuously differentiable functions of τ when $\tau=\tau_2$. So the singularity of simultaneous binary collision in equation (3.11) is removed by transferring to equation (3.20)-(3.25). This concludes the proof of theorem 3.2. which yields the proof of theorem 3.1. $\sharp$

3.1.2 C^2 Regularization of the Simultaneous Binary Collisions

Furthermore, we even can prove that the regularization is C^2 in theorem 3.3.

Theorem 3.3. *The equations (3.20)-(3.25) give rise to a C^2 extension of $\vec{p}_2(\tau)$ with respect to τ at $\vec{p}_2(\tau_2)$ the simultaneous binary collision.*

Proof. Let $F(\tau)=\frac{\xi_1(\tau)}{\xi_2(\tau)}$. Then $F(\tau_2)=\alpha^{1/2}$. From equations (3.20)-(3.25) and lemma 3.2, we have at $\tau=\tau_2$,

$$\frac{d\xi_1}{d\tau}=(1+\alpha^{-1})(-2\sqrt{m_1+m2}),\ \frac{d\xi_2}{d\tau}=(1+\alpha)(-2\sqrt{m_3+m_4}),\ \frac{d\eta_1}{d\tau}=0,\ \frac{d\eta_2}{d\tau}=0.$$

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$$\lim_{\tau\to\tau_2}\frac{dF}{d\tau}=\lim_{\tau\to\tau_2}\frac{\frac{d\xi_1}{d\tau}\xi_2-\xi_1\frac{d\xi_2}{d\tau}}{\xi_2^2}=\lim_{\tau\to\tau_2}(1+F^{-2})\frac{\eta_1-F^3\eta_2}{\xi_2}$$

$$=(1+\alpha^{-2})\lim_{\tau\to\tau_2}\frac{\frac{d\eta_1}{d\tau}-F^3\frac{d\eta_2}{d\tau}-3F^2\eta_2\frac{dF}{d\tau}}{d\xi_2}=-3\alpha^{-1/2}\lim_{\tau\to\tau_2}\frac{dF}{d\tau}$$

So $\lim_{\tau\to\tau_2}\frac{dF}{d\tau}=0$ and

$$\lim_{\tau\to\tau_2}\frac{d^2\xi_1}{d\tau^2}=\lim_{\tau\to\tau_2}\frac{d((1+F^{-2})\eta_1)}{d\tau}=\lim_{\tau\to\tau_2}(1+F^{-2})\frac{d\eta_1}{d\tau}+\lim_{\tau\to\tau_2}(1-2F^{-3}\frac{dF}{d\tau})\eta_1=-2\sqrt{m_1+m_2}.$$

Similarly, we can prove the limits of the second derivative of $\xi_1(\tau),\xi_2(\tau),\xi_3(\tau),\eta_1(\tau),\eta_2(\tau),\eta_3(\tau)$ exist at $\tau=\tau_2$. Therefore the second derivatives are continuously differentiable functions of τ when $\tau=\tau_2$. The extension of the simultaneous collision orbit is C^2. $\sharp$

3.2 Periodic Solutions with SBC

Let us recall our notation. x_1,x_2,x_3,x_4 are the positions of collinear four body problem with the center of mass at origin, i.e. (3.4) holds. $u_1=x_2-x_1$ is the difference of the first two bodies and $u_2=x_4-x_3$ is the difference of the last two bodies. $u_3=\frac{m_1x_1+m_2x_2}{m_1+m_2}$ is the center of mass of the first two bodies. v_i are the derivatives corresponding to u_i, $i=1,2,3$. Furthermore, we have a new coordinates and time scale given by (3.18) and (3.19). $\bigwedge=\{x_1=x_2,x_3=x_4,x_2\neq x_3\}=\{u_1=u_2=0,u_3<0\}=\{\xi_1=\xi_2=0,\xi_3\neq 0\}$ are the sets of simultaneous binary collision in the respective coordinates. In this section we always assume that $x=(x_1,x_2,x_3,x_4)$ be a simultaneous binary collision solution which is defined in $t\in[t_1,t_2)$ encountering with singular set $\bigwedge$ at $t=t_2$. $\vec{p}_1(t)=(u_1(t),u_2(t),u_3(t),v_1(t),v_2(t),v_3(t))$ is obtained by transformation (3.6) and $\vec{p}_2(\tau)=(\xi_1(\tau),\xi_2(\tau),\xi_3(\tau),\eta_1(\tau),\eta_2(\tau),\eta_3(\tau))$ is obtained by transformation (3.18) and new time scale (3.18). By theorem 3.2 and theorem 3.3, $\vec{p}_2(\tau)$ is a C^2 solution of equation (3.20)-(3.25) without singularity at $\tau=\tau_2$. Furthermore, there exist $\tau_4>\tau_2$, such that the behavior of the extension of $\vec{p}_2(\tau)$ can be described by time reverse in (τ_2,τ_4) as follows.

3.2.1 Time Reverse Extension of Simultaneous Binary Collision

Theorem 3.4. *Suppose that $\vec{p}_2(\tau)=(\xi_1,\xi_2,\xi_3,\eta_1,\eta_2,\eta_3)$ is defined as a simultaneous binary collision solution of (3.20)-(3.25) in (τ_1,τ_2) and is extended to (τ_1,τ_3) in theorem 3.2, where*

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$\tau_1 < \tau_2 < \tau_3$. *Let* $\vec{p}_3(\tau) = (\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3, \tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3)$ *denote as follows,*

$$\tilde{\xi}_1(\tau) = \begin{cases} \xi_1(\tau) & \tau_1 < \tau \leq \tau_2, \\ -\xi_1(2\tau_2 - \tau) & \tau_2 < \tau < \tau_4. \end{cases} \tag{3.33}$$

$$\tilde{\xi}_2(\tau) = \begin{cases} \xi_2(\tau) & \tau_1 < \tau \leq \tau_2, \\ -\xi_2(2\tau_2 - \tau) & \tau_2 < \tau < \tau_4. \end{cases} \tag{3.34}$$

$$\tilde{\xi}_3(\tau) = \begin{cases} \xi_3(\tau) & \tau_1 < \tau \leq \tau_2, \\ \xi_3(2\tau_2 - \tau) & \tau_2 < \tau < \tau_4. \end{cases} \tag{3.35}$$

$$\tilde{\eta}_1(\tau) = \begin{cases} \eta_1(\tau) & \tau_1 < \tau \leq \tau_2, \\ \eta_1(2\tau_2 - \tau) & \tau_2 < \tau < \tau_4. \end{cases} \tag{3.36}$$

$$\tilde{\eta}_2(\tau) = \begin{cases} \eta_2(\tau) & \tau_1 < \tau \leq \tau_2, \\ \eta_2(2\tau_2 - \tau) & \tau_2 < \tau < \tau_4. \end{cases} \tag{3.37}$$

$$\tilde{\eta}_3(\tau) = \begin{cases} \eta_3(\tau) & \tau_1 < \tau \leq \tau_2, \\ -\eta_3(2\tau_2 - \tau) & \tau_2 < \tau < \tau_4. \end{cases} \tag{3.38}$$

where $\tau_4 = \min\{2\tau_2 - \tau_1, \tau_3\}$. *Then* $\vec{p}_3(\tau) = \vec{p}_2(\tau)$ *for* $\tau \in (\tau_1, \tau_4)$.

Proof. We only need to verify that the extension $\vec{p}_3(\tau)$ of $\vec{p}_2(\tau)$ also satisfies the differential equations (3.20)-(3.25) in (τ_1, τ_4) and the smoothness $\vec{p}_3(\tau)$ at $\tau = \tau_2$.

For $\tau_1 < \tau \leq \tau_2$, $\vec{p}_3(\tau) = \vec{p}_2(\tau)$ then $\vec{p}_3(\tau)$ obviously satisfies the differential equations (3.20)-(3.25). For $\tau_2 < \tau < \tau_4$,

$$\frac{d\tilde{\xi}_1}{d\tau} = \frac{d(-\xi_1(2\tau_2 - \tau))}{d\tau} = \frac{d\xi_1}{d\tau}\Big|_{2\tau_2 - \tau}$$

$$= \frac{(\xi_1(2\tau_2 - \tau))^2 + (\xi_2^2(2\tau_2 - \tau))^2}{(\xi_1(2\tau_2 - \tau))^2}\eta_1(2\tau_2 - \tau) = \frac{\tilde{\xi}_1^2 + \tilde{\xi}_2^2}{\tilde{\xi}_1^2}\tilde{\eta}_1.$$

Because $\xi_1(\tau_2) = 0$, the time reverse extension $\tilde{\xi}_1$ of ξ_1 is continuously differentiable at τ_2. ξ_1 across the singular set because the derivative of ξ_1 at τ_2 is not zero. In fact, it is negative. The figure 11 illustrates the extension for ξ_1.

For $\tau_2 < \tau < \tau_4$, because $\frac{d\eta_1}{d\tau}|_{\tau_2} = 0$ and

$$\frac{d\tilde{\eta}_1}{d\tau} = \frac{d(\eta_1(2\tau_2 - \tau))}{d\tau} = -\frac{d\eta_1}{d\tau}|_{2\tau_2 - \tau}$$

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$$= \frac{(\tilde{\eta}_1^2 - 4(m_1+m_2))}{\tilde{\xi}_1^2}\frac{\tilde{\xi}_1(\tilde{\xi}_1^2+\tilde{\xi}_2^2)}{\tilde{\xi}_1^2} + \left(m_3(K_{32}^2 - K_{31}^2) + m_4(K_{42}^2 - K_{41}^2)\right)\tilde{\xi}_1(\tilde{\xi}_1^2+\tilde{\xi}_2^2),$$

where K_{ij} only involve square terms ξ_i^2, the time reverse extension $\tilde{\eta}_1$ of η_1 also satisfies the differential equation (3.23). The figure 12 is an example of $\tilde{\eta}_1$.

The proof of extension $\tilde{\xi}_2, \tilde{\eta}_3$ and $\tilde{\xi}_3, \tilde{\eta}_2$ is similar to the proof of extension $\tilde{\xi}_1$ and $\tilde{\eta}_1$ respectively.

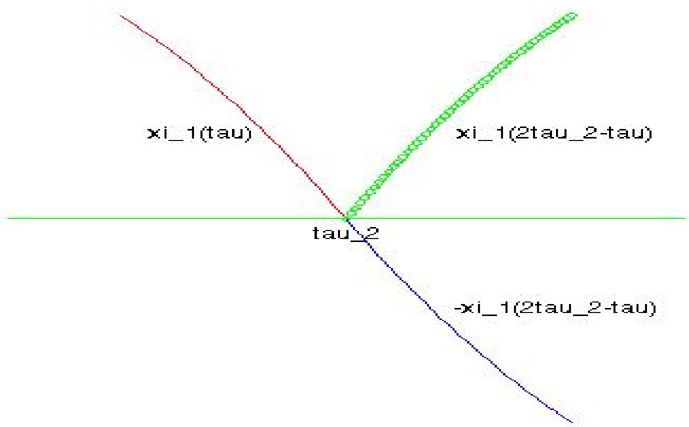

Figure 11: Extension of ξ_1 to $\tilde{\xi}_1$

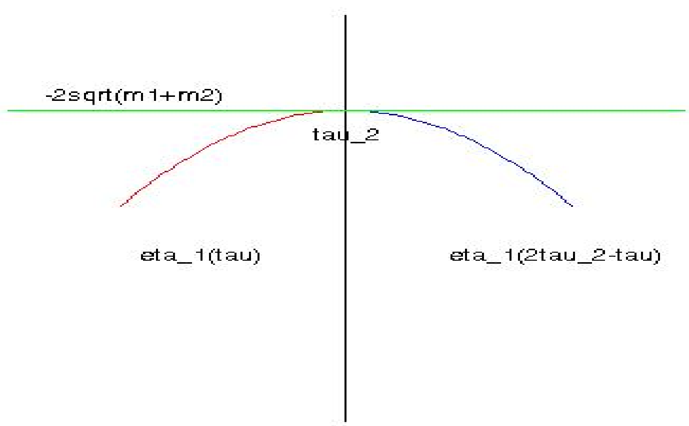

Figure 12: Extension of η_1 to $\tilde{\eta}_1$

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So $\vec{p}_3(\tau)$ is also a solution of differential equation (3.20)-(3.25) and it is the same as $\vec{p}_2(\tau)$ when $\tau \leq \tau_2$. By the uniqueness theorem of ordinary differential equations, $\vec{p}_3(\tau)$ must equal $\vec{p}_2(\tau)$ for $\tau \in (\tau_1, \tau_4)$. $\sharp$

3.2.2 Behavior of SBC at the Singular Set

Now we are going to describe the behavior of the simultaneous binary collision solution when it is closing to and at the singular set $\bigwedge$ in the original coordinate.

Theorem 3.5. *Let $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ be the extended simultaneous binary collision solution of (3.1) by converting the C^2 solution $\vec{p}_3(\tau)$ into original system. Then the solution $x(t)$ has the following properties.*

(a) x is defined in $t \in [t_1, t_4]$, where $t_4 = t(\tau_4)$ and $t_1 < t_2 < t_4 \leq 2t_2 - t_1$. x encounters with singular set $\bigwedge$ when $t = t_2$.

(b) Let $C_1 = \frac{m_1x_1+m_2x_2}{m_1+m_2}$ be the center of mass of m_1, m_2 and $C_2 = \frac{m_3x_3+m_4x_4}{m_3+m_4}$ be the center of mass of m_3, m_4. Then

$$\lim_{t\to t_2} \frac{C_1}{C_2} = -\frac{m_3+m_4}{m_1+m_2},$$

where $C_1 < 0$ and $C_2 > 0$ are both finite.

(c) The ratio of velocity $\frac{dx_i}{dt}$ and $\frac{dx_{i+1}}{dt}$ approaches a finite number as $t \to t_2$, where $i = 1, 3$, more precisely, $\lim_{t\to t_2} \frac{dx_i}{dt} / \frac{dx_{i+1}}{dt} = -\frac{m_{i+1}}{m_i}$. The negative sign implies that the velocities of collision pairs are in opposite direction as $t \to t_2$, which is independent of the initial positions and the initial velocities.

(d) The ratio of the distance $u_1 = x_2 - x_1$ and the distance $u_2 = x_4 - x_3$ is determined by their mass ratio, more precisely, $\lim_{t\to t_2} \frac{u_1}{u_2} = \left(\frac{m_1+m_2}{m_3+m_4}\right)^{\frac{1}{3}}$. The ratio of their velocities is also determined by their mass ratio, i.e. $\lim_{t\to t_2} \frac{du_1/dt}{du_2/dt} = \left(\frac{m_1+m_2}{m_3+m_4}\right)^{\frac{1}{3}}$.

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(e) In the original time scale, the velocities are unbounded, i.e., $\lim_{t\to t_2}\frac{dx_i}{dt}=\infty$. *But in the new time scale, the velocities are bounded and* $\lim_{\tau\to\tau_2}\frac{dx_i}{d\tau}=0$, *where* $i=1,\cdots,4$.

Proof. (a) is directly from the theorem 3.2 and theorem 3.4. By making using of the center of mass at origin, it is easy to prove (b). (c) is directly from the equation (3.1) and L'Hopital's rule, in fact,

$$\lim_{t\to t_2}\frac{\frac{dx_1}{dt}}{\frac{dx_2}{dt}}=\lim_{t\to t_2}\frac{\frac{d^2x_1}{dt^2}}{\frac{d^2x_2}{dt^2}}=\lim_{t\to t_2}\frac{\sum_{j\neq 1}\frac{m_j(x_j-x_1)}{|x_j-x_1|^3}}{\sum_{j\neq 2}\frac{m_j(x_j-x_2)}{|x_j-x_2|^3}}=-\frac{m_2}{m_1},$$

and $\lim_{t\to t_2}\frac{\frac{dx_3}{dt}}{\frac{dx_4}{dt}}=-\frac{m_4}{m_3}$. (d) is directly from lemma 3.1. Now we turn to prove (e).
In the new coordinates and new time scale, as the solution $\vec{p}_2(\tau)$ approaches the singular set $\bigwedge$, we already have, from the proof of theorem 3.2 and theorem 3.4,

$$\xi_1(\tau_2)=0,\xi_2(\tau_2)=0,\xi_3(\tau_2)<0,\eta_1(\tau_2)=-2\sqrt{m_1+m_2},\eta_2(\tau_2)=-2\sqrt{m_3+m_4}$$

and $\eta_3(\tau_2)$ is finite. So in the new time scale, it slows down the motion to a finite speed (η_i are related to the velocity of the particles). Recall that $\frac{du_1}{dt}\to\infty$ as t goes to t_2, but $u_1=\frac{\xi_1^2}{2}$ implies

$$\frac{du_1}{d\tau}=\xi_1\frac{d\xi_1}{d\tau}=0 \text{ at } \tau=\tau_2.$$

From which we have,

$$\frac{d(x_2-x_1)}{d\tau}=\frac{dx_2}{d\tau}-\frac{dx_1}{d\tau}=0 \text{ at } \tau=\tau_2.$$

From 2 above, we have

$$\lim_{\tau\to\tau_2}\frac{\frac{dx_2}{d\tau}}{\frac{dx_1}{d\tau}}=\lim_{t\to t_2}\frac{\frac{dx_2}{dt}\frac{dt}{d\tau}}{\frac{dx_1}{dt}\frac{dt}{d\tau}}=\lim_{t\to t_2}\frac{\frac{dx_2}{dt}}{\frac{dx_1}{dt}}=-\frac{m_2}{m_1}$$

Therefore,

$$\frac{dx_2}{d\tau}=0 \text{ and } \frac{dx_1}{d\tau}=0 \text{ at } \tau=\tau_2.$$

$$\eta_3(\tau)=\xi_3(\tau)v_3(\tau)=\xi_3(\tau)\frac{m_1\frac{dx_1}{d\tau}+m_2\frac{dx_2}{d\tau}}{m_1+m_2}=0 \text{ at } \tau=\tau_2.$$

We complete the proof of theorem 3.5. $\sharp$

3.2.3 Construction of Periodic Solutions

Using the above properties in subsection 3.2.2, we construct a family of periodic solutions of the collinear four body problem with simultaneous binary collisions.

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Let $x^0 = (x_1^0, x_2^0, x_3^0, x_4^0)$ denote the initial positions of collinear four body problem with $-\infty < x_1^0 < x_2^0 < 0 < x_3^0 < x_4^0 < \infty$. We assume that x^0 possesses symmetries on positions and masses, i.e. $x_1^0 = -x_4^0, x_2^0 = -x_3^0$ and $m_1 = m_4, m_2 = m_3$. Without loss of generality, let $s = x_2^0 - x_1^0 = x_4^0 - x_3^0 > 0$, $x_2^0 = -1, x_3^0 = 1$, and $m_1 = m_4 = m, m_2 = m_3 = 1$

Theorem 3.6. *If s and m fall into the region of $\frac{s^2(s+2)^2}{16(1+s)} < m$ in the first quadrant of sm-plane (see Figure 13), then the orbit by releasing the four bodies with zero velocity at x^0 is a periodic orbit with simultaneous binary collisions .*

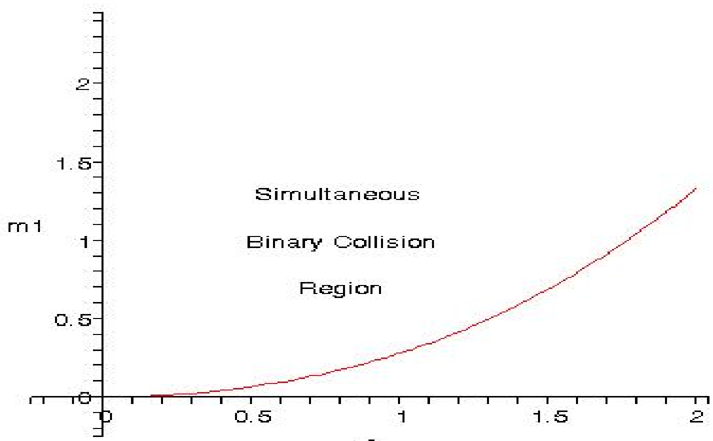

Figure 13: SBC Region

Before we prove theorem 3.6, we prove the following lemma. Let $x = (x_1, x_2, x_3, x_4)$ denote the positions of our four particles on the line with positive mass (m_1, m_2, m_3, m_4). We assume, without loss of generality, that $x_1 \leq x_2 \leq x_3 \leq x_4$ and the center of mass is at origin.

Lemma 3.3 (Periodic Solution with Simultaneous Binary Collision) *Let $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ be a smooth solution of (3.1) in the interval $[t_0, t_2)$ with initial condition $x(0) = x^0 = (x_1^0, x_2^0, x_3^0, x_4^0)$ and $\frac{dx}{dt}(0) = 0$, where $t_2 > t_0 = 0$. If the solution $x(t)$ has a simultaneous binary collision at $t_2 = T > 0$, then the solution $x(t)$ can be extended to a periodic solution with period*

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$2T$ as follows, in the sense of regularization given by theorem 3.2,

$$\tilde{x}(t) = \begin{cases} x(t-2nT) & 2nT \le t \le (2n+1)T, \\ x((2n+2)T-t) & (2n+1)T \le t \le (2n+2)T. \end{cases} \tag{3.39}$$

Proof. Because the motion is governed by Newton's differential equation (3.1), it encounters a singularity at $t_2 = T$ which the velocities of the bodies approaching collision go to infinity. So the equation can not give information of the motion in a neighborhood of t_2. But the singularity at $t_2 = T$ causing by simultaneous binary collision can be removed in the sense of theorem 3.2. Then the orbit can be obtained in the following steps.

Step 1: The four particles are released at $t_0 = 0$ with initial positions $x^0 = x(t_0) = (x_1^0, x_2^0, x_3^0, x_4^0)$ and zero initial velocity. During time interval $(t_0, t_1]$, $t_1 < t_2$, the motion of the four particles are described by Newton's differential equation (3.1) (see Figure 14).

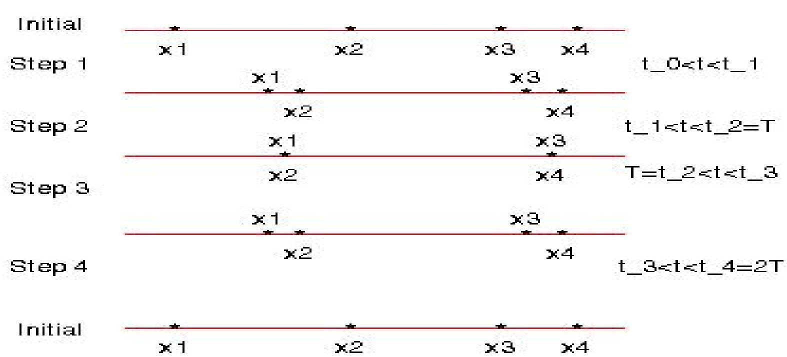

Figure 14: Periodic Solution with SBC

There is no any collision in the time interval $(t_0, t_1]$ and $x(t_1) = (x_1^1, x_2^1, x_3^1, x_4^1)$ close to simultaneous binary collision, i.e. $0 < x_2^1 - x_1^1 < \rho, -\delta < \frac{m_1 x_1^1 + m_2 x_2^1}{m_1 + m_2} < -\delta^{-1}$.

Step 2: Because $x(t_1)$ falls into $\mathcal{U}_{\rho,\delta}$ and leads to a simultaneous binary collision solution, the motion of x can be described by the differential equations (3.20)-(3.25) in the new coordinates (3.18) and the new time scale (3.19). During the time (t_1, t_2), x approaches a simultaneous binary collision and encountering the singular set $\bigwedge$ at $t = t_2 = T$.

Step 3: By theorem 3.4, the motion can be extended as a time reverse, i.e. $\tilde{x}(t) = x(2T - t)$

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for $t \in (t_2, t_3), t_3 = 2t_2 - t_1$

Step 4: The position $x(t_3)$ of the particles at t_3 is equal to the position $x(t_1)$ but their velocity is just opposite by step 3. Then in the following time interval $(t_3, t_4], t_4 = 2T$, the particles go back to the initial position at t_4 and they have zero velocity at t_4. The motion in $(t_3, t_4]$ is described by equations (3.1).

Then the orbit completes one period in $[0, 2T]$ and it repeats step 1, step 2, step 3, step 4. So the solution is extended to a periodic solution with simultaneous collision at $t = (2n+1)T$, where n is an integer.

Note that the time scale is τ and new coordinates are ξ_i and η_i, $i = 1, 2, 3$ in step 2 and step 3. But we can change back to x and t by (3.18) and (3.19). ♯

The proof of theorem 3.6. We only need check whether the conditions in theorem 3.6 lead to a simultaneous binary collision without any other collisions. By Newton's law, the accelerations of the four particles are respectively,

$$a_1 = m_2 s^{-2} + m_3(s+2)^{-2} + m_4(2s+2)^{-2},$$

$$a_2 = -m_1 s^{-2} + \frac{m_3}{4} + m_4(s+2)^{-2},$$

$$a_3 = -m_1(s+2)^{-2} - \frac{m_2}{4} + m_4 s^{-2},$$

$$a_4 = -m_1(2s+2)^{-2} - m_2(s+2)^{-2} - m_3 s^{-2}.$$

Note that $m_1 = m_4 = m, m_3 = m_2 = 1$ then no matter the choice of s and m, $a_1 > 0$ and $a_4 < 0$. If s, m can be chosen such that the acceleration $a_2 < 0$ but $a_3 > 0$, then x_0 leads to a simultaneous binary solution if it is released with zero velocity because of the symmetry of positions and masses. Therefore it is extended to a periodic solution with singularity.

In order that $a_2 < 0$ and $a_3 > 0$, we only need make $a_2 < 0$ by choosing proper s, m_1 because $a_2 = -a_3$. The numerator of a_2 is

$$na_2 = -16m_1 s - 16m_1 + s^4 + 4s^3 + 4s^2,$$

and the denominator of a_2 is

$$da_2 = 4s^2(s+2)^2.$$

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So when s, m_1 fall into the region of $\frac{s^2(s^2+4s+4)}{16(1+s)} < m$, it has $a_2 < 0$ and leads to a simultaneous binary collision (see Figure 14). $\sharp$

3.3 Periodic Solutions with Single Binary Collisions and Simultaneous Binary Collisions

The behavior of the motion for the pair closing the single binary collision can be described as time reverse plus a higher order term in a very short time neighborhood. At the moment of single binary collision, the velocities of the particles involving the collision approach to infinity. By changing the time scale, the velocities of the particles remain bounded as slow motion. The motion can be extended to cross the collision point. Any periodic solution of collinear four body problem involves collisions. In this section many periodic solutions are constructed with both a sequence of single binary collisions and a sequence of simultaneous binary collisions in the collinear four body problem. The central configuration of collinear four body problem plays an important role in our construction. It separates periodic solutions with collisions into two categories: type A periodic solution (not involving single binary collisions), and type B periodic solution (involving single binary collisions).

Let $x^0 = (x_1^0, x_2^0, x_3^0, x_4^0)$ denote the initial positions of collinear four body problem with $-\infty < x_1^0 < x_2^0 < 0 < x_3^0 < x_4^0 < \infty$. We assume that x^0 possesses symmetries on positions and masses, i.e., $x_1^0 = -x_4^0, x_2^0 = -x_3^0$ and $m_1 = m_4, m_2 = m_3$. Without loss of generality, let $s_0 = x_2^0 - x_1^0 = x_4^0 - x_3^0 > 0$, $x_2^0 = -1, x_3^0 = 1$, and $m_1 = m_4 = m, m_2 = m_3 = 1$.

$x^0 = (-s_0 - 1, -1, 1, s_0 + 1)$ with mass $(m, 1, 1, m)$ forms a central configuration if and only if

$$m = \frac{(s_0+1)^2\left({s_0}^5 + 5\,{s_0}^4 + 8\,{s_0}^3 - 4\,{s_0}^2 - 16\,s_0 - 16\right)}{17\,{s_0}^4 + 68\,{s_0}^3 + 100\,{s_0}^2 + 64\,s_0 + 16}. \tag{3.40}$$

For $s_0 > 1.396812289$, there is a positive $m > 0$ such that $x^0 = (-s_0 - 1, -1, 1, s_0 + 1)$ with mass $(m, 1, 1, m)$ forms a central configuration. The result is a special case of theorem 2.1 in subsection 2.2. In this section, all the motion are obtained by releasing the four bodies at initial position with zero velocity.

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3.3.1 Type A Periodic Solution

Theorem 3.7. *Assume that* $x^0 = (-s_0 - 1, -1, 1, s_0 + 1)$ *with mass* $(m, 1, 1, m)$ *forms a central configuration. Then* $y^0 = (-s_0 + s - 1, -1, 1, s_0 - s + 1)$ *with mass* $(m, 1, 1, m)$ *leads to a periodic solution only involving simultaneous binary collision if the four bodies are released with zero velocity, where* $0 < s < s_0$.

Proof. It is well known that there is a unique central configuration with the fixed order of four given masses in collinear four body problem. Because $x^0 = (-s_0 - 1, -1, 1, s_0 + 1)$ with mass $(m, 1, 1, m)$ form a central configuration (the formula of central configuration for s_0 and m is given as above), then $y^0 = (-s_0 + s - 1, -1, 1, s_0 - s + 1)$ with mass $(m, 1, 1, m)$ can not lead to a total collision with $0 < s < s_0$. By symmetry, y^0 only can lead to either a single binary collision first between m_2 and m_3 or a simultaneous binary collision first. By lemma 3.3, if y^0 leads to a simultaneous binary collision first, then $y^0 = (-s_0 + s - 1, -1, 1, s_0 - s + 1)$ leads to a periodic solution only involving simultaneous binary collision.

Claim:For $0 < s < s_0$, y^0 can not lead to a single binary collision between m_2 and m_3.

At x^0, the accelerations of the four particles are respectively,

$$ax_1 = m_2 s_0^{-2} + m_3(s_0 + 2)^{-2} + m_4(2s_0 + 2)^{-2},$$

$$ax_2 = -m_1 s_0^{-2} + \frac{m_3}{4} + m_4(s_0 + 2)^{-2},$$

$$ax_3 = -m_1(s_0 + 2)^{-2} - \frac{m_2}{4} + m_4 s_0^{-2},$$

$$ax_4 = -m_1(2s_0 + 2)^{-2} - m_2(s_0 + 2)^{-2} - m_3 s_0^{-2}.$$

They lead to a total collision. When the initial condition changes to $y^0 = (-s_0 + s - 1, -1, 1, s_0 - s + 1)$, the accelerations of the four particles are respectively,

$$ay_1 = m_2(-s_0 + s)^{-2} + m_3(s - s_0 - 2)^{-2} + m_4(2(s_0 - s) + 2)^{-2},$$

$$ay_2 = -m_1(-s_0 + s)^{-2} + \frac{m_3}{4} + m_4((s - s_0 - 2)^{-2},$$

$$ay_3 = -m_1(s - s_0 - 2)^{-2} - \frac{m_2}{4} + m_4(-s_0 + s)^{-2},$$

$$ay_4 = -m_1(2(s_0 - s) + 2)^{-2} - m_2(s_0 + 2)^{-2} - m_3 s_0^{-2}.$$

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By direct computation, it is easy to see $0 < ax_1 < ay_1$ but $ax_2 > ay_2$ and symmetrically for other two bodies. This implies that m_1 and m_2 shall collide before m_2 and m_3 collides by comparing this motion with the motion having total collision. So the motion with initial position y^0 and zero initial velocity can not have a single binary collision first between m_2 and m_3. By symmetry, it must lead to a simultaneous binary collision. Therefore, by lemma 3.3, it leads to a periodic solution only involving simultaneous binary collision. Figure 15 illustrates a case that m_2 and m_3 move inside first (but not collide) then turn back to a simultaneous binary collision.

3.3.2 Type B Periodic Solution

Lemma 3.4 *Let $x^0 = (-s_0 - 1, -1, 1, s_0 + 1)$ with mass $(m, 1, 1, m)$ form a central configuration, where $s_0 > 0$. Then there exist a unique $s_1^* > 0$, such that, for any $0 < s < s_1^*$, $y^0 = (-s_0 - s - 1, -1, 1, s_0 + s + 1)$ leads to a periodic solution involving exact one single binary collision between m_2 and m_3 before a simultaneous binary collision in $[0, T]$, where T corresponds to the first time of simultaneous binary collision after releasing the four bodies from y^0 at time $t = 0$ with zero velocity.*

Proof. Claim 1: For small $s > 0$, the motion with initial position y^0 and zero velocity at $t = 0$ has exact one single binary collision between m_2 and m_3 before a simultaneous binary collision in $[0, T]$.

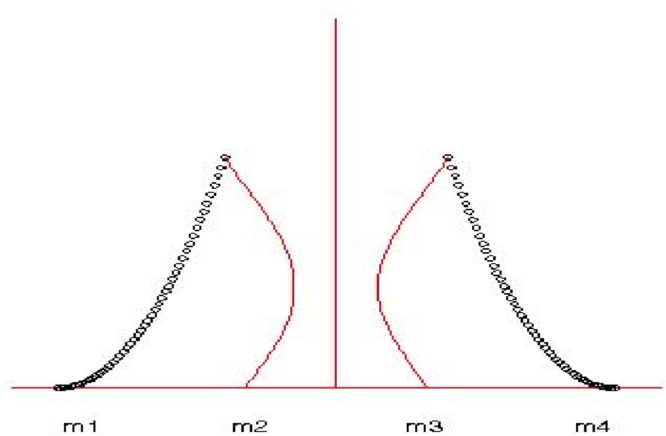

Figure 15: Type A Periodic Solution

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Proof of the claim 1. By similar argument as in the proof of theorem 3.7, the accelerations of m_1 and m_2 at time $t = 0$ satisfy $0 < ay_1 < ax_1$ but $ax_2 < ay_2$ and symmetrically for other two bodies. This implies that m_2 and m_3 shall collide at origin before m_1 and m_2 collides by comparing the motion with total collision. So the motion with initial position $y^0 = (-s_0 - s - 1, -1, 1, s_0 + s + 1)$ has a single binary collision between m_2 and m_3 first at time $0 < t_1 < T$. It can be regularized and then the motion will continue and keep its symmetry. m_1 continues to right and m_2 bounds back.

If m_1 and m_2 don't collide after t_1, then there exists a time t_2 with $t_1 < t_2$ such that m_2 turns back to right, that is, the velocity of m_2 at t_2 is zero. Let $y(t) = (y_1(t), y_2(t), y_3(t), y_4(t))$ be the solution with initial position y^0 and zero initial velocity. Comparing the orbit of $y(t)$ in $[0, t_1)$ with the orbit of $y(t')$ in $(t_1, t_2]$, we shall have $y_2(0) = y_2(t_2) < 0$ if there is no force on m_2 from m_1 and m_4. But when the position $y_2(t)$ of m_2 in $[0, t_1)$ is equal to the position $y_2(t')$ of m_2 in $[t_1, t_2)$, $\ddot{y}_2(t) > \ddot{y}_2(t') > 0$ because m_1 and m_4 are closer to m_2 at t' than at t. Therefore we have $y_2(t_2) < y_2(0) < 0$. Similarly, $0 < y_3(0) < y_3(t_2)$.

If s is small enough, m_1 can go over position $y_2(0)$ at time t_1, i.e. $-1 < y_1(t_1) < 0$. So m_1 and m_2 must collide, say at time T, after the single binary collision between m_2 and m_3 by continuity argument and by comparing with total collision. Then the collision must be a simultaneous binary collision by symmetry. The orbit can be extended to a periodic solution with exact one single binary collision (at t_1) and one simultaneous binary collision (at T) in $[0, T]$.

Claim 2: There exist a $\tilde{s} > 0$, such that the motion with initial position $y^0 = (-s_0 - \tilde{s} - 1, -1, 1, s_0 + \tilde{s} + 1)$ has at least two single binary collision between m_2 and m_3 before m_1 and m_4 are involved in any collisions.

Proof of claim 2. Consider an auxiliary system $z = (-r, z_2, z_3, r)$ with mass $(m, 1, 1, m)$ by fixing $z_1 = -r, z_4 = r$ under Newton's law. So $z_1(t) = -r, z_2(t) = r$ and z_2, z_3 are determined by the following equations,

$$\ddot{z}_2(t) = -\frac{m}{(z_2 + r)^2} + \frac{1}{(z_2 - z_3)^2} + \frac{m}{(z_2 - r)^2} \tag{3.41}$$

$$\ddot{z}_3(t) = -\frac{m}{(z_3 + r)^2} - \frac{1}{(z_3 - z_2)^2} + \frac{m}{(z_3 - r)^2} \tag{3.42}$$

with initial position $z(0) = (-r, -1, 1, r)$ and zero initial velocity. By the symmetry of differential

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equations and initial conditions, $z_3(t) = -z_2(t)$.

In fact, equation (3.41) is a Hamiltonian system with

$$H = \frac{1}{2}|\dot{z}_2|^2 + \left(-\frac{m}{(z_2+r)} + \frac{1}{4z_2} + \frac{m}{(z_2-r)}\right).$$

H is a constant along solution $z_2(t)$. At $t = 0$, $z_2(0) = -1, \dot{z}_2(0) = 0$, then $H \equiv C = \left(-\frac{m}{(-1+r)} - \frac{1}{4} + \frac{m}{(-1-r)}\right) = -(\frac{2mr}{r^2-1} + \frac{1}{4}) < 0$. Assume that $z_2(t)$ travels from -1 to 0 in $[0, t_1]$. We have

$$dt = \frac{dz_2}{\sqrt{2\left(C - \left(-\frac{m}{(z_2+r)} + \frac{1}{4z_2} + \frac{m}{(z_2-r)}\right)\right)}}.$$

So

$$\begin{aligned}
t_1(m,r) &= \int_{-1}^{0} \frac{dz_2}{\sqrt{2\left(C - \left(-\frac{m}{(z_2+r)} + \frac{1}{4z_2} + \frac{m}{(z_2-r)}\right)\right)}} \\
&= \int_{-1}^{0} \frac{dz_2}{\sqrt{2\left(C - \frac{2mr}{(z_2^2-r^2)} - \frac{1}{4z_2}\right)}} \\
&\geq \int_{-1}^{0} \frac{dz_2}{\sqrt{2\left(C - \frac{2mr}{((-1)^2-r^2)} - \frac{1}{4z_2}\right)}} \\
&= \int_{-1}^{0} \frac{dz_2}{\sqrt{2\left(-\frac{1}{4} - \frac{1}{4z_2}\right)}} = \frac{\sqrt{2}\pi}{2}
\end{aligned}$$

From (3.41), the acceleration of m_2 can be always positive if r is large, in fact,

$$\begin{aligned}
\ddot{z}_2(t) &= -\frac{m}{(z_2+r)^2} + \frac{1}{(z_2-z_3)^2} + \frac{m}{(z_2-r)^2} \\
&= \frac{2mrz_2}{(z_2^2-r^2)^2} + \frac{1}{4z_2^2} \geq \frac{-2mr}{(1-r^2)^2} + \frac{1}{4z_2^2} > 0
\end{aligned}$$

if $m < \frac{(1-r^2)^2}{8r}$ and $z_2 \in [-1, 0]$. Then we have

$$t_1(m,r) \leq \int_{-1}^{0} \frac{dz_2}{\sqrt{2\left(C - \frac{2mrz_2}{(1-r^2)} - \frac{1}{4z_2}\right)}} < \infty. \tag{3.43}$$

The above integral is the time that the z_2 moves from -1 to 0 with the smaller acceleration $\frac{-2mr}{(1-r^2)^2} + \frac{1}{4z_2^2} > 0$. For any finite time T' with $3t_1(m,r) < T' < \infty$, there exists a large r such that m_2 and m_3 collide at origin at least two times in the finite time interval $[0, T']$.

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Consider another auxiliary system $w = (w_1, -r, r, w_4)$ with mass $(m, 1, 1, m)$ by fixing $w_2 = -r, w_3 = r$ under Newton's law. So w_1, w_4 are determined by the following equations,

$$\ddot{w}_1(t) = \frac{1}{(w_1+r)^2} + \frac{1}{(w_1-r)^2} + \frac{m}{(w_1-w_4)^2} \tag{3.44}$$

$$\ddot{w}_4(t) = -\frac{1}{(w_4-r)^2} - \frac{1}{(w_4+r)^2} - \frac{m}{(w_4-w_1)^2} \tag{3.45}$$

with initial positions $(-\tilde{s}, -r, r, \tilde{s})$ and zero initial velocity. Within the motion of w_1 from $\tilde{s}$ to $-r$, we have

$$\ddot{w}_1(t) \leq \frac{k}{(w_1+r)^2}$$

because $(w_1+r)^2 < (w_1-r)^2 < 4w_1^2$, where $k = \max\{1, m\}$. By the similar argument as $t_1(m, r)$, the time t_2 for the motion of m_1 from $-\tilde{s}$ to $-r$ has

$$t_2(m, \tilde{s}) \geq \int_{-\tilde{s}}^{-r} \frac{dw_1}{\sqrt{2(-C_2 - \frac{k}{w_1+r})}} = \frac{k\pi}{(2C_2)^{3/2}} = \frac{\pi}{2\sqrt{2k}}(\tilde{s}-r)^{3/2}, \tag{3.46}$$

where $C_2 = \frac{k}{\tilde{s}-r}$. After comparing (3.43) with (3.46), we know that if $\tilde{s}$ is large enough, w_1 can not across $-r$ and w_4 can not across r in the finite time $[0, T']$.

Now for the large $\tilde{s}$, the motion $y(t) = (y_1(t), y_2(t), y_3(t), y_4(t))$ with initial position $y^0 = (-s_0 - \tilde{s} - 1, -1, 1, s_0 + \tilde{s} + 1)$ and zero initial velocity has the following properties. (1) In the finite time T', y_1 can not across $-r$ and y_4 can not across r by comparing with the auxiliary system $w(t)$ because $0 < \ddot{y}_1(t) < \ddot{w}_1(t)$. and $0 > \ddot{y}_4(t) > \ddot{w}_4(t)$. (2) In the finite time T', y_2 and y_3 should collide at least two times by comparing with the auxiliary system $z(t)$.

Claim 3: If the motion with initial position $y^1 = (-s_0 - s_1 - 1, -1, 1, s_0 + s_1 + 1)$ and zero initial velocity has n times single binary collision between m_2 and m_3 in [0,T], where T corresponds to the first time of simultaneous binary collision, then $y^2 = (-s_0 - s_2 - 1, -1, 1, s_0 + s_2 + 1)$ leads to at least n times single binary collision before simultaneous binary collision for $0 < s_1 < s_2$ in $[0, T]$.

Proof of Claim 3. At time $t = 0$, $\ddot{y}_1^1 > \ddot{y}_1^2 > 0$ but $0 < \ddot{y}_2^1 < \ddot{y}_2^2$, this implies that y_1^2 goes slower than y_1^1 but y_2^2 goes faster than y_1^2. Therefore, y^2 takes shorter time to have the first single binary collision between m_2 and m_3. By similar argument, we can prove that y^2 also take shorter time to have the second binary collision between m_2 and m_3. Then in $[0, T]$, y^2 has at leat n times single binary collision before simultaneous binary collision.

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Now assume that $s_1^* = \sup\{s > 0 : y^0 = (-s_0 - s - 1, -1, 1, s_0 + s + 1)$ *leads to a periodic solution involving only one single binary collision between* m_2 *and* m_3 *and one simultaneous binary collision in one period.* $\}$

The claim 1 proves the existence of s_1^*. The claim 2 and claim 3 prove s_1^* is finite and unique. This completes the proof of lemma 3.4.♯

Theorem 3.8. *Let* $x^0 = (-s_0-1, -1, 1, s_0+1)$ *with mass* $(m, 1, 1, m)$ *form a central configuration, where* $s_0 > 0$ *is implicitly defined in (3.40).*
(1) Then there exists a sequence $0 < s_1^* < s_2^* < \cdots$ *such that the motion has exact* n *times single binary collision between* m_2 *and* m_3 *before a simultaneous binary collision in* $[0, T]$*, where* T *corresponds to the first time of simultaneous binary collision, if the motion starts with initial position* $y^0 = (-s_0 - s_n - 1, -1, 1, s_0 + s_n + 1)$ *and zero initial velocity, where* $s_{n-1}^* < s_n < s_n^*$.
(2)If four particles are released from $y^0 = (-s_0 - s_n^* - 1, -1, 1, s_0 + s_n^* + 1)$ *with zero initial velocity,* $n = 1, 2, \cdots$ *, then the motion ends at a total collision after* n *times single binary collision between* m_2 *and* m_3.

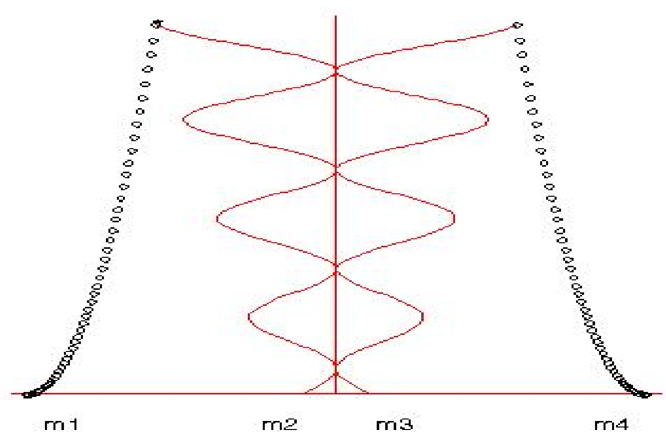

Figure 16: Type B Periodic Solution

Proof. The proof can be done by induction base on the proof of lemma 3.4. Figure 16 illustrates an example that m_2 and m_3 collide 4 times before a simultaneous binary collision.

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Remark: Let $x^0 = (-s_0 - 1, -1, 1, s_0 + 1)$ with mass $(m, 1, 1, m)$ form a central configuration, where $s_0 > 0$. Assume $y^0 = (-s_0 - s_n - 1, -1, 1, s_0 + s_n + 1)$ denote the initial position of the collinear four bodies, where $s^*_{n-1} < s_n < s^*_n$. Let $y(t) = (y_1(t), y_2(t), y_3(t), y_4(t))$ be the periodic solutions involving single binary collisions and simultaneous binary collisions with $y(0) = y^0$ and $\frac{dy}{dt}(0) = 0$. Then there exist a time sequence $0 = t_1 < t_2 < \cdots < t_n < T$ such that $y_2(t_{i+1}) \leq y_2(t_i) < 0, 0 < y_3(t_i) \leq y_3(t_{i+1})$, and $\frac{dy_j}{dt}(t_i) = 0$, where $j = 2, 3, i = 1, \cdots, n-1$. Figure 16 illustrates an example that m_3 goes further each time.

4 Stability of Periodic Solutions Generated from Central Configuration

In 1772, Lagrange discovered his remarkable equilateral periodic solutions of the planar three-body problem [13]. For any choice of the three masses, there exists a family of periodic solutions, each body travelling along a specific Kepler orbit. Contained in the family are two types of periodic orbits: rigid circular motion (choosing a circular Kepler orbit) and homographic motion (choosing an elliptic Kepler orbit).

A crucial first step in analyzing the local behavior near a periodic solution is to compute the characteristic multipliers of the linearized equations. For the circular case, this was first accomplished by Gascheau in 1834 in his thesis [19]. Recently Roberts [40] in 2002 showed that the stability of the family of periodic solution depends on two parameters– the eccentricity e of the orbit and the mass parameter $\beta = 27(m_1 m_2 + m_1 m_3 + m_2 m_3)/(m_1 + m_2 + m_3)^2$. Roberts was able to reduce the dimensions of the problem from 12 to 4 by eliminating 8 standard first integrals and then making a clever change of coordinates. By analyzing the behavior of the characteristic multipliers and how they vary with e and β, he eventually obtained the region of stability and instability of the Kepler periodic solution.

As mentioned in section 2, for $n \geq 4$ it is very difficult to find all the central configurations,

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much less to analyze their stability. The exceptions are the highly symmetrical central configurations, like the regular polygon with equal masses. Some progress has been made in finding and analyzing the stability of central configurations of the four-body problem [4],[16],[46].

In this section, we study the stability of Kepler orbits for rhombus four body problem. First, we carefully reduce the dimensions of the problem from 16 to 4. This is achieved by means of symmetry and by eliminating the standard integrals. Then we make a change of coordinates which decouples the associated linear system. One of the resulting systems yields two +1 multipliers, expected due to the nature of the periodic solution. The other system is two dimensional and governs the linear instability of the periodic solution. This system is a type of Hill Equation. The resulting system is marvellously simple and only depends on the size of the rhombus and the eccentricity e. We then analyze the behavior of the characteristic multipliers and how they vary with e and the size of the rhombus. We prove all the Kepler periodic solutions of the rhombus four body problem are unstable.

4.1 Kepler Orbits

The N-body problem configuration $q = (q_1, q_2, \cdots, q_N)$ describes a planar positions of N point masses $m_1, \cdots, m_N$, where $q_i \in \mathbb{R}^2$. Newtonian system (1.1) is the second-order ordinary differential equation system:

$$m_i \ddot{q}_i = \sum_{i \neq j} \frac{m_i m_j (q_j - q_i)}{|q_j - q_i|^3} = \frac{\partial U}{\partial q_j}, \qquad i = 1, 2, \cdots, N. \tag{4.1}$$

where the function

$$U(q) = \sum_{1 \leq k < j \leq n} \frac{m_k m_j}{|q_k - q_j|}.$$

is called the potential function on the set of noncollision configurations (where $q_i \neq q_j, i \neq j$). The Hamiltonian for the N-body problem is the difference of kinetic minus potential

$$H(q_1, q_2, \cdots, q_N, p_1, p_2, \cdots, p_N) = \sum_{i}^{N} \frac{1}{2m_i} \|p_i\|^2 - U(q).$$

The Hamiltonian equations of the N-body problem are

$$\dot{q}_i = \frac{1}{m_i} p_i, \qquad \dot{p}_i = \frac{\partial U(q)}{\partial q_i}, \qquad i = 1, 2, \cdots, N \tag{4.2}$$

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We recall the fact that the N-body problem always admits *uniformly rotating* solutions which generalize the circular rotational solutions of the Kepler equation. Following the presentation in [22], we are looking for solutions of the form

$$q_i(t) = \psi(t) q_{0i}, \qquad\qquad i = 1, 2, \cdots, N \tag{4.3}$$

where $\psi(t)$ is a scalar function and q_{0i} is a constant vector. For the moment, identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ so that $q_i(t), \psi(t), q_{0i}$ are complex numbers. Substituting this guess (4.3) into equation (4.1), we have

$$|\psi|^3 \psi^{-1} \ddot{\psi} m_i q_{0i} = \sum_{i \neq j} \frac{m_i m_j (q_{0i} - q_{0j})}{\|q_{0i} - q_{0j}\|^3}.$$

This can be split into an equation for the scalar function $\psi(t)$

$$\ddot{\psi} = -\frac{\mu \psi}{|\psi|^3} \tag{4.4}$$

and an equation for the initial positions $\mathbf{q_0} = (q_{01}, \cdots, q_{0N})$

$$\sum_{i \neq j} \frac{m_i m_j (q_{0i} - q_{0j})}{|q_{0i} - q_{0j}|^3} + \mu m_i q_{0i} = 0. \tag{4.5}$$

The motion of our special solution is determined by equation (4.4), which is simply the planar Kepler Problem. Among the solutions to this problem are periodic orbits on circles and ellipses. The initial shape of the solution in position space $\mathbb{R}^{2N}$ is determined by equation (4.5) and the solution is called a central configuration (see Definition 2.1). The above analysis shows that all central configurations $\mathbf{q}_0$ admit *homothetic solutions* $\mathbf{q}(t) = \psi(t)\mathbf{q}_0$, $\psi(t) \in \mathbb{R}$ satisfies (4.4). Such homothetic solutions end in total collapse. Ejection orbits are the time reversal of collision orbits.

Coplanar central configurations admit in addition homographic solutions $\mathbf{q}(t) = \psi(t)\mathbf{q}_0$, $\psi(t) \in \mathbb{C}$ where each of the N-masses executes a similar keplerian ellipse of eccentricity $e, 0 \leq e \leq 1$. When $e = 1$ the homographic solutions degenerate to a homothetic solution which includes total collapse. When $e = 0$, the relative equilibrium solutions are recovered consisting of uniform circular motion for each of the masses about the common center of mass.

Consider the rhombus four body problem, let $\mathbf{q}_0 = (q_{01}, \cdots, q_{04})$ be the position vector as shown in Figure 17.

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Note that summing equation (4.5) over all i, one obtain $\sum_{i=1}^{4} m_i q_{0i} = 0$ so that the center of mass is at the origin. For simplicity, we choose $q_{01} = (-a, 0)$ with $a > 0$ and $q_{03} = (0, b)$ with $b > 0$. Similarly by the symmetry of rhombus, the other two coordinates q_{02}, q_{04} are also determined. Once given a, b, and $\frac{1}{\sqrt{3}}a < b < \sqrt{3}a$, Long and Ouyang [16] proved that $m_2 = m_1, m_4 = m_3$ and the masses are determined by the configuration. Furthermore, we can scale the masses so that the parameter $\mu = 1$.

This fixes a pair of unique values of m_1, m_3 as a function of the two parameters a, b. By checking equation (4.5), we find

$$m_1 = 4\,\frac{\left(a^2+b^2\right)^{3/2}\left(8\,b^3-\left(a^2+b^2\right)^{3/2}\right)a^3}{64\,a^3b^3-\left(a^2+b^2\right)^3} \tag{4.6}$$

$$m_3 = 4\,\frac{\left(8\,a^3-\left(a^2+b^2\right)^{3/2}\right)b^3\left(a^2+b^2\right)^{3/2}}{64\,a^3b^3-\left(a^2+b^2\right)^3} \tag{4.7}$$

Kepler's equation (4.4) is solvable up to quadrature [22] page 100. In polar coordinates (r, θ), the solution with $\mu = 1$ is given by

$$r(t) = \frac{\omega^2}{1 + e\cos\theta(t)}, \qquad \dot{\theta} = \frac{\omega}{r^2}, \qquad \theta(0) = 0, \tag{4.8}$$

where e, the eccentricity of the ellipse, and ω, the angular momentum, are two parameters. We have chosen the argument of the perihelion and $\theta(0)$ both to be zero. This means the true anomaly begins at zero and is measured from the positive horizontal axis. While these choices clearly do

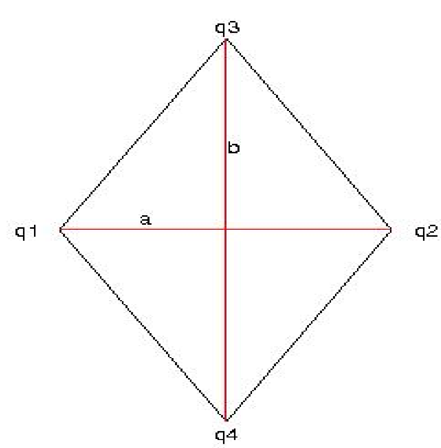

Figure 17: Rhombus Four Body Problem

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not affect the stability of the periodic orbits, the parameters e and ω could. But we will show that ω has no effect on the linear stability of kepler's orbits.

If we write our central configuration in polar coordinates, $q_{0i} = \bar{r}_i(\cos\bar{\theta}_i, \sin\bar{\theta}_i)$, where $\bar{r}_1 = \bar{r}_2 = a$, $\bar{r}_3 = \bar{r}_4 = b$ and $\bar{\theta}_1 = \pi, \bar{\theta}_2 = 2\pi, \bar{\theta}_3 = \frac{\pi}{2}, \bar{\theta}_4 = \frac{3\pi}{2}$ then the position component of the periodic orbit is written as

$$q_i(t) = \bar{r}_i r(t)(\cos(\theta(t) + \bar{\theta}_i), \sin(\theta(t) + \bar{\theta}_i)). \tag{4.9}$$

In order to study the stability of the periodic solution, one has to compute a fundamental matrix solution $X(t)$ to the equations of motion linearized about the periodic orbit. The monodromy matrix is the matrix C satisfying $X(t + T) = X(t)C$ (for example see [5]). Stability is governed by the eigenvalues of the monodromy matrix, called the characteristic multipliers. Since we are dealing with a Hamiltonian system, C is symplectic and the multipliers are symmetric about the unit circle. In order to have linear stability, it is necessary that all the multipliers have modulus one. For the planar four body problem, it is a 16 dimensional ODE system (4.2). As is well known, the N-body problem is a Hamiltonian system with several first integrals, therefore we can reduce the dimension by eliminating all 8 standard first integrals. But the remaining system still has 8 dimensions even after eliminating all first integrals. For this reason, it is still hard to analyze the stability of the Kepler periodic orbits.

Here we employ a new approach to reduce the degree of freedom of the Hamilton system by means of symmetry constraint. The constrained Hamilton system has 2 degrees of freedom and the corresponding ODE system which is a type of Hill equation has dimension 2 after eliminating the first integrals. The instability of the original system is governed by the two dimensional ODE system. However the stability of the new ODE system may not corresponds to the stability of the original ODE system. But we will show the Kepler orbits of rhombus four body problem are unstable in the reduced system, therefore, Kepler orbits of rhombus four body problem are unstable in original system.

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4.2 Constrained Hamilton System on Rhombus Four-Body Problem

Let us turn to the variational method [24] to construct the constrained Hamilton system on rhombus four-body problem. We define the *Lagrangian* $L(\mathbf{q}(t), \dot{\mathbf{q}}(t)) = L(q_1(t), \cdots, q_4(t), \dot{q}_1(t), \cdots, \dot{q}_4(t))$ to be the Kinetic energy *minus* the potential energy of the system and $\dot{q}_i = \frac{dq}{dt}$ to be the velocity:

$$L(q_1(t), \cdots, q_4(t), \dot{q}_1(t), \cdots, \dot{q}_4(t)) = \sum_{i=1}^{4} \frac{m_i}{2} |\dot{q}_i|^2 + \sum_{i<j} \frac{m_i m_j}{|q_i - q_j|}.$$

Then the action functional

$$I[\mathbf{q}(t)] := \int_0^T \sum_{i=1}^{4} \frac{m_i}{2} |\dot{q}_i|^2 + \sum_{i<j} \frac{m_i m_j}{|q_i - q_j|} dt, \qquad \mathbf{q}(t) \in M$$

is defined for absolutely continuous T-periodic curves $\mathbf{q}(t)$ in the configuration manifold $M = \{\mathbf{q}(t) \in C^2([0,T]; \mathbb{R}^{2\times 4}) | \mathbf{q}(t+T) = \mathbf{q}(t)\}$. For the particular rhombus four body problem, we will look for *symmetric solutions* of the equations of the motion. Consider the following symmetric function space in polar coordinates

$$M = \left\{ \begin{array}{l} q_1(t) = r_1(t) \exp(i\theta_1), q_2(t) = \exp(i\pi) q_1(t), \\ q_3(t) = \frac{r_3(t)}{r_1(t)} \exp(-i\frac{\pi}{2}) q_1(t), q_4(t) = \exp(i\pi) q_3(t) \end{array} \right\}.$$

Under these constraints, $\dot{q}_1 = (\dot{r}_1 \cos(\theta_1) - r_1 \sin(\theta_1)\dot{\theta}_1, \dot{r}_1 \sin(\theta_1) + \dot{r}_1 \cos(\theta_1)\dot{\theta}_1)$, we have a new Lagrangian in polar coordinates

$$L(r_1, r_3, \theta_1, \dot{r}_1, \dot{r}_3, \dot{\theta}_1) = \frac{1}{2}[m_1(\dot{r}_1^2 + r_1^2\dot{\theta}_1^2) + m_2(\dot{r}_1^2 + r_1^2\dot{\theta}_1^2) + m_3(\dot{r}_3^2 + r_3^2\dot{\theta}_1^2) + m_4(\dot{r}_3^2 + r_3^2\dot{\theta}_1^2)]$$
$$+ \frac{m_1 m_2}{2r_1} + \frac{m_1 m_3}{k} + \frac{m_1 m_4}{k} + \frac{m_2 m_3}{k} + \frac{m_2 m_4}{k} + \frac{m_3 m_4}{2r_3},$$

where $k = \sqrt{r_1^2 + r_3^2}$. Then the corresponding conjugate variables R_1, R_3, Θ_1 with respect to r_1, r_3, θ_1 are

$$R_1 = (m_1 + m_2)\dot{r}_1, R_3 = (m_3 + m_4)\dot{r}_3, \Theta_1 = (m_1 r_1^2 + m_2 r_1^2 + m_3 r_3^2 + m_4 r_3^2)\dot{\theta}_1.$$

The new Lagrangian is a projection of the 12-dimensional Euler-Lagrange flow on a non-invariant 6-dimensional submanifold (the tangent space of the space of rhombi). The Hamiltonian is

$$H_1 = \frac{{R_1}^2}{2(m_1 + m_2)} + \frac{{R_3}^2}{2(m_3 + m_4)} + \frac{{\Theta_1}^2}{2(m_1\, {r_1}^2 + m_2\, {r_1}^2 + m_3\, {r_3}^2 + m_4\, {r_3}^2)} \tag{4.10}$$
$$- \frac{m_1\, m_2}{2r_1} - \frac{m_1\, m_3}{k} - \frac{m_1\, m_4}{k} - \frac{m_2\, m_3}{k} - \frac{m_2\, m_4}{k} - \frac{m_3\, m_4}{2r_3}.$$

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The Hamiltonian will be independent of θ_1 which means that Θ_1 (angular momentum) is a first integral, and θ_1 is an ignorable variable. Setting $\Theta_1 = c$ and substituting into equation (4.10) gives

$$H = \frac{{R_1}^2}{2(m_1+m_2)} + \frac{{R_3}^2}{2(m_3+m_4)} + \frac{c^2}{2(m_1\,{r_1}^2 + m_2\,{r_1}^2 + m_3\,{r_3}^2 + m_4\,{r_3}^2)} \tag{4.11}$$
$$-\frac{m_1\,m_2}{2r_1} - \frac{m_1\,m_3}{k} - \frac{m_1\,m_4}{k} - \frac{m_2\,m_3}{k} - \frac{m_2\,m_4}{k} - \frac{m_3\,m_4}{2r_3}.$$

This reduces the system to four dimensions, in the variables (r_1, r_3, R_1, R_3). The equations of motion in these new variables are

$$\dot{r}_1 = \frac{R_1}{m_1+m_2},$$
$$\dot{r}_3 = \frac{R_3}{m_3+m_4},$$
$$\dot{R}_1 = \frac{c^2\,(m_1\,r_1 + m_2\,r_1)}{(m_1\,{r_1}^2 + m_2\,{r_1}^2 + m_3\,{r_3}^2 + m_4\,{r_3}^2)^2} - \frac{m_1\,m_2}{2{r_1}^2} - \frac{m_1\,m_3\,r_1}{({r_1}^2+{r_3}^2)^{3/2}}$$
$$-\frac{m_1\,m_4\,r_1}{({r_1}^2+{r_3}^2)^{3/2}} - \frac{m_2\,m_3\,r_1}{({r_1}^2+{r_3}^2)^{3/2}} - \frac{m_2\,m_4\,r_1}{({r_1}^2+{r_3}^2)^{3/2}},$$
$$\dot{R}_3 = \frac{c^2\,(\,m_3\,r_3 + \,m_4\,r_3)}{(m_1\,{r_1}^2 + m_2\,{r_1}^2 + m_3\,{r_3}^2 + m_4\,{r_3}^2)^2} - \frac{m_1\,m_3\,r_3}{({r_1}^2+{r_3}^2)^{3/2}} - \frac{m_1\,m_4\,r_3}{({r_1}^2+{r_3}^2)^{3/2}}$$
$$-\frac{m_2\,m_3\,r_3}{({r_1}^2+{r_3}^2)^{3/2}} - \frac{m_2\,m_4\,r_3}{({r_1}^2+{r_3}^2)^{3/2}} - \frac{m_3\,m_4}{2{r_3}^2}.$$

Although the submanifold M is non-invariant, we still have lemma 4.1.

Lemma 4.1. If $\gamma_1(t) = (q_1(t), \cdots, q_4(t), p_1(t), \cdots, p_4(t))$ is a critical point of the original system and $\gamma_1(t) = (q_1(t), \cdots, q_4(t), p_1(t), \cdots, p_4(t))$ is also in the constrained space M, then $\gamma_2(t) = (r_1(t), r_3(t), R_1(t), R_3(t))$ is a periodic solution of the constrained Hamiltonian system, where $\gamma_2(t)$ is from $\gamma_1(t)$ by relation M.

Proof. By the construction of the new Lagrangian, it is easy to prove Lemma 4.1. The following solution is an example. $\sharp$

Using (4.8) and (4.9), a short calculation shows that the Kepler periodic solution, denoted in general as $\gamma(t)$, is written as

$$r_1(t) = ar(t), \qquad R_1(t) = (m_1+m_2)aR(t),$$
$$r_3(t) = br(t), \qquad R_3(t) = (m_3+m_4)bR(t) \tag{4.12}$$

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where $m_2 = m_1, m_4 = m_3$ and m_1, m_3 satisfy the equations (4.6), (4.7). Recall that

$$r(t) = \frac{\omega^2}{1 + e\cos\theta(t)}, \qquad \dot{r}(t) = R(t), \qquad \dot{\theta} = \frac{\omega}{r^2}, \qquad \theta(0) = 0$$

is the periodic solution to Kepler's problem mentioned earlier. In addition to the size of the rhombus, the two parameters in this solution are eccentricity e and the angular momentum ω of the elliptic orbits. The total angular momentum for the full problem has the value $\Theta_1 = c = (m_1 a^2 + m_2 a^2 + m_3 b^2 + m_4 b^2)\omega$.

Lemma 4.2. If the Kepler solution $\gamma_2(t)$ is linearly unstable in constrained Hamiltonian system, then the Kepler solution $\gamma_1(t)$ is also linearly unstable in original Hamiltonian system, where $\gamma_2(t)$ is constructed from $\gamma_1(t)$ by relation M.

Proof. We'll prove it by contradiction. Assume $\gamma_1(t)$ is linearly stable in original Hamiltonian system. For any initial condition v_{10} with $|v_{10}|$ very small, the solution $v_1(t, v_{10})$ of the original linearized Hamiltonian system along $\gamma_1(t)$ is also very small, where $v_1(0, v_{10}) = v_{10}$. Because the constrained linearized Hamiltonian system along $\gamma_2(t)$ is the projection of 12 dimensional original system, for any initial condition v_{20} with $|v_{20}|$ very small, the solution $v_2(t, v_{20})$ of the constrained linearized Hamiltonian system along $\gamma_2(t)$ is also very small, where $v_2(0, v_{20}) = v_{20}$. So $\gamma_2(t)$ is also stable. $\sharp$

In order to study the linear stability of the Kepler solution $\gamma(t)$ given by (4.12), we will linearize the reduced Hamilton equation along the Kepler solution. Then we compute a fundamental matrix solution $X(t)$ to the linearized equation and calculate its eigenvalues. But we will decouple the system before we compute the fundamental solution.

Linearizing the four-dimensional system about the periodic solution $\gamma(t)$ gives the time-dependent periodic linear Hamiltonian system

$$\dot{X}(t) = J_2 D^2 H(\gamma(t))X = A(t)X$$

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where $A(t) = J_2 D^2 H(\gamma(t))$ and J_2 is the canonical matrix

$$J_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

After a good deal of calculation and simplification, we have

$$A(t) = \begin{bmatrix} 0 & 0 & \frac{1}{2m_1} & 0 \\ 0 & 0 & 0 & \frac{1}{2m_3} \\ D_{31} & D_{32} & 0 & 0 \\ D_{32} & D_{42} & 0 & 0 \end{bmatrix}$$

where

$$D_{31} = \frac{-8\omega^2 {m_1}^2 a^2}{r^4\,(m_1\,a^2 + m_3\,b^2)} + \frac{2\omega^2 m_1}{r^4} + \frac{{m_1}^2}{r^3 a^3} + \frac{12 m_1\, m_3\, a^2}{(a^2+b^2)^{5/2}\, r^3} - \frac{4 m_1\, m_3}{(a^2+b^2)^{3/2}\, r^3},$$

$$D_{32} = \frac{-8\omega^2 m_3\, b m_1\, a}{r^4\,(m_1\,a^2 + m_3\,b^2)} + \frac{12 m_1\, m_3\, ba}{(a^2+b^2)^{5/2}\, r^3},$$

$$D_{42} = \frac{-8\omega^2 {m_3}^2 b^2}{r^4\,(m_1\,a^2 + m_3\,b^2)} + \frac{2\omega^2 m_3}{r^4} + \frac{12 m_1\, m_3\, b^2}{(a^2+b^2)^{5/2}\, r^3} - \frac{4 m_1\, m_3}{(a^2+b^2)^{3/2}\, r^3} + \frac{{m_3}^2}{r^3 b^3}.$$

4.3 Decoupling the Linear System

In this subsection, we follow the Gareth E. Roberts' ideas and presentation in [40] to decouple the linear system. For the convenience of the reader we give a complete proof and develop the method to decouple the linear system.

A linear, time-dependent periodic Hamiltonian system is one of the form

$$\dot{X}(t) = J D^2 H(\gamma(t)) X \tag{4.13}$$

where J is the canonical matrix

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

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and $D^2H(t+T) = D^2H(t)$.

When such a system results from linearizing about a periodic solution, it can be shown that there are at least two $+1$ characteristic multipliers. One of these is attributable to the periodic orbit and another arise from the existence of an integral, which in this case is the Hamiltonian H. This fact is easily proved via differentiation [5], [22]. Indeed, given a periodic solution $\gamma(t)$ to a Hamiltonian system $\dot{x} = J\nabla H(x)$, plugging in $\gamma(t)$ and differentiating with respect to t yields

$$\ddot{\gamma}(t) = J_2 D^2 H(\gamma(t))\dot{\gamma}(t). \tag{4.14}$$

Thus, $\dot{\gamma}(t)$ is a solution of the associated linear system. Since $\gamma(t)$ is periodic, so is its derivative. If we choose coordinates so that $\gamma(0) = (1, 0, \cdots, 0)$, the first column of the monodromy matrix is $(1, 0, \cdots, 0)$ and $+1$ is an eigenvalue. But relation (4.14) is important for another reason. That is, it suggests a useful change of coordinates. Choosing variables so that the periodic orbit is easily represented helps decouple the system. This follows from a standard result in the theory of Hamiltonian systems.

Define the skew-inner product of two vectors $v, w \in \mathbb{C}^{4n}$ as

$$\Omega(v, w) = v^T J w.$$

Note that $J^T = -J = J^{-1}$ so that J is orthogonal and skew-symmetric. A key trait of linear Hamiltonian systems is that the skew-orthogonal complement of an invariant subspace is also invariant.

Lemma 4.3. Suppose W is an invariant subspace of the matrix $JD^2H(t)$, then the skew-orthogonal complement of W, defined as $W^\perp = \{v \in \mathbb{C}^{4b} : \Omega(v, w) = 0 \ \forall w \in W\}$, is also an invariant subspace of $JD^2H(t)$.

Proof. Suppose $v \in W^\perp$. Then, for any $w \in W$ we have

$$\Omega(JD^2H(t)v, w) = v^T D^2H(t) J^T J w = v^T D^2 H(t) w = -v^T J\hat{w} = 0,$$

where $\hat{w} = JD^2H(t)w \in W$. Thus $JD^2H(t)v \in W^\perp$. $\sharp$

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Given an invariant subspace, Lemma 4.3 shows that a simple linear change of variables will decouple the system. The characteristic multipliers remain the same since the transformation is linear. To apply these ideas to our problem, we need to find an invariant subspace for $A(t) = J_2 D^2 H(\gamma(t))$. As mentioned before, the periodic orbit itself provides an excellent suggestion. We make use of the fact that the Kepler periodic solution $r(t)$ satisfies

$$\ddot{r}(t) = \frac{\omega^2}{r^3} - \frac{1}{r^2}. \tag{4.15}$$

$$\dddot{r}(t) = \left(-\frac{3\omega^2}{r^4} + \frac{2}{r^3}\right)\dot{r}. \tag{4.16}$$

From these we have

$$\gamma = \begin{bmatrix} r_1 \\ r_3 \\ R_1 \\ R_3 \end{bmatrix} = \begin{bmatrix} ar(t) \\ br(t) \\ (m_1 + m_2)aR(t) \\ (m_3 + m_4)bR(t) \end{bmatrix} = \begin{bmatrix} ar(t) \\ br(t) \\ (m_1 + m_2)a\dot{r}(t) \\ (m_3 + m_4)b\dot{r}(t) \end{bmatrix},$$

and

$$\dot{\gamma} = \begin{bmatrix} a\dot{r}(t) \\ b\dot{r}(t) \\ (m_1 + m_2)a\ddot{r}(t) \\ (m_3 + m_4)b\ddot{r}(t) \end{bmatrix} \quad \text{and} \quad \ddot{\gamma} = \begin{bmatrix} a\ddot{r}(t) \\ b\ddot{r}(t) \\ (m_1 + m_2)a\left(-\frac{3\omega^2}{r^4} + \frac{2}{r^3}\right)\dot{r}(t) \\ (m_3 + m_4)b\left(-\frac{3\omega^2}{r^4} + \frac{2}{r^3}\right)\dot{r}(t) \end{bmatrix},$$

as expressions for the first and second derivatives of the periodic orbit. A short calculation gives

$$A(t)\dot{\gamma}(t) = J_2 D^2 H(\gamma(t))\dot{\gamma}(t) = \ddot{\gamma}(t),$$

$$A(t)\ddot{\gamma}(t) = J_2 D^2 H(\gamma(t))\ddot{\gamma}(t) = \left(-\frac{3\omega^2}{r^4} + \frac{2}{r^3}\right)\dot{\gamma}(t).$$

Then the vectors $W_1 := [a, b, 0, 0], W_2 := [0, 0, 2\, m_1\, a, 2\, m_3\, b]$ will span an invariant subspace W for $J_2 D^2 H(\gamma(t))$. We then apply the equalities $m_2 = m_1, m_4 = m_3$ as well as other relations such as (4.6) to (4.8) as needed. Consider the change of variables determined by

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$$\begin{pmatrix} r_1 \\ r_3 \\ R_1 \\ R_3 \end{pmatrix} = \begin{bmatrix} a & 0 & 2\,m_3\,b & 0 \\ b & 0 & -2\,m_1\,a & 0 \\ 0 & 2\,m_1\,a & 0 & b \\ 0 & 2\,m_3\,b & 0 & -a \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

The determinant of the linear transformation matrix is $-4\,{m_1}^2a^4 - 8\,m_3\,b^2m_1\,a^2 - 4\,{m_3}^2b^4$ which is nonzero. The last two columns of the above matrix are chosen to form a basis for the skew-orthogonal complement of W. Consequently, this change of variables will decouple our linear system into two 2×2 system. The new coordinates are

$$\begin{aligned} x_1 &= \frac{m_1\,ar_1}{m_1\,a^2 + m_3\,b^2} + \frac{m_3\,br_3}{m_1\,a^2 + m_3\,b^2}, \\ x_2 &= \frac{aR_1}{2(m_1\,a^2 + m_3\,b^2)} + \frac{bR_3}{2(m_1\,a^2 + m_3\,b^2)}, \\ x_3 &= \frac{br_1}{2(m_1\,a^2 + m_3\,b^2)} - \frac{ar_3}{2(m_1\,a^2 + m_3\,b^2)}, \\ x_4 &= \frac{m_3\,bR_1}{m_1\,a^2 + m_3\,b^2} - \frac{m_1\,aR_3}{m_1\,a^2 + m_3\,b^2}, \end{aligned}$$

and the new differential equation system is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \left(-\frac{3\omega^2}{r^4} + \frac{2}{r^3}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{34} \\ 0 & 0 & D_{43} & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix},$$

where $D_{34} = \frac{1}{4m_1\,m_3}$, and

$$D_{43} = -\frac{8\left(a^4{m_1}^2 - 2\,a^2{m_1}^2b^2 + 6\,a^2m_3\,b^2m_1 - 2\,a^2{m_3}^2b^2 + {m_3}^2b^4\right)m_1\,m_3}{r^3\left(m_1\,a^2 + m_3\,b^2\right)\left(a^2+b^2\right)^{5/2}} + \frac{4m_1\,m_3\,\omega^2}{r^4} + \frac{2{m_3}^2{m_1}^2\left(a^5+b^5\right)}{r^3b^3\left(m_1\,a^2 + m_3\,b^2\right)a^3}.$$

Note that along the periodic orbit $\gamma(t)$, $x_1 = r, x_2 = R, x_3 = x_4 = 0$. Thus, we expect the first 2×2 system in the x_1, x_2 variables to identify the two $+1$ multipliers, leaving the remaining two variables x_3, x_4 to decide the linear instability of the Kepler solution.

The equations for the x_1 and x_2 variables give a simple 2×2 periodic, linear Hamiltonian system:

$$\dot{x}_1 = x_2,$$

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$$\dot{x}_2 = \left(-\frac{3\omega^2}{r^4} + \frac{2}{r^3}\right) x_1.$$

For the initial condition $x_1(0) = 0, x_2(0) = 1$, making use of (4.15), (4.16), we have as a solution $x_1 = h\dot{r}, x_2 = h\ddot{r}$, where $h = \omega^4/(e(1+e)^2)$ is chosen so that $x_2(0) = 1$. Since this is a periodic solution with the same period as the system itself, the second column of the monodromy matrix for this system will be (0,1). Since we have a Hamiltonian system, the monodromy matrix is symplectic, with determinant one, and must have the form

$$\begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix}.$$

The equations for the remaining two variables give the following 2×2 periodic, linear Hamiltonian system:

$$\dot{x}_3 = D_{34} x_4,$$

$$\dot{x}_4 = D_{43} x_3.$$

We now make a scaling of the variables using the transformation $\hat{x}_3 = \omega^{-3/2} x_3, \hat{x}_4 = \omega^{3/2} x_4$. Since this is a linear transformation, it will not change the characteristic multipliers. Next, we change the independent variable from t to θ. In other words, use

$$\dot{x}_3 = \frac{dx_3}{dt} = \frac{dx_3}{d\theta}\frac{d\theta}{dt} = x_3' \frac{\omega}{r^2}$$

and similar expressions for $\dot{x}_4$ and $\dot{r}$. Dropping the hats off the variables and letting $'$ represent the derivative with respect to θ, our final two-dimensional system for the linearization about the relative periodic orbit $\gamma(t)$ is

$$\begin{bmatrix} x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 0 & f(a,b,e,\theta) \\ g(a,b,e,\theta) & 0 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \tag{4.17}$$

where $f(a,b,e,\theta) = \frac{1}{4 m_3\, m_1\, (1+e\cos(\theta))^2}$ and

$$g(a,b,e,\theta) = 4 m_3\, m_1\, (1+e\cos(\theta))^2 + \frac{2 {m_3}^2 {m_1}^2 \left(a^5+b^5\right)(1+e\cos(\theta))}{b^3\,(m_1\, a^2 + m_3\, b^2)\, a^3}$$

$$-\frac{8\left(a^4 {m_1}^2 - 2\, a^2 {m_1}^2 b^2 + 6\, a^2 m_3\, b^2 m_1 - 2\, a^2 {m_3}^2 b^2 + {m_3}^2 b^4\right) m_1\, m_3\, (1+e\cos(\theta))}{(m_1\, a^2 + m_3\, b^2)\,(a^2+b^2)^{5/2}}.$$

m_1, m_3 satisfy the equation (4.6), (4.7). The differential equation system (4.17) could be regarded as Hill's equation. One crucial fact about this system is that the masses are determined by the

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parameters a, b of the rhombus size through (4.6), (4.7). Moreover, since ω is not present, the angular momentum of the elliptic Kepler orbit does not affect the linear stability. So the stability depends on a, b and e the eccentricity of the elliptic Kepler orbit.

4.4 Linear Stability Analysis

In this section, we use a standard method [5] to analyze the linear stability of the elliptic Kepler orbits in terms of the parameters a, b and e through system (4.17). This is done by computing the eigenvalues of Monodromy matrix. This method was used in [40] and we apply it to our problem. Since system (4.17) has been derived from a Hamiltonian system , the characteristic polynomial of M will be reciprocal [22]. In other words, λ is an eigenvalue of M if and only if $1/\lambda$ is also an eigenvalue. Therefore in order to have linear stability, all the eigenvalues must be on the unit circle. Then our main theorem in this section reads as:

Theorem 4.1. *For all possible values of the three parameters a, b and e, the elliptic periodic orbits of rhombus four-body problem are linearly unstable.*

Proof. The characteristic polynomial of M has the form

$$p(\lambda) = det(\lambda I - M) = \lambda^2 + q\lambda + 1, \tag{4.18}$$

where $q = -\text{tr}(M)$ and $det(M) = 1$.

Given that the multipliers are on the unit circle, there are two ways in which stability can be lost:

1. period-doubling bifurcation (two -1 eigenvalues), occurring when $q = 2$,
2. two +1 eigenvalues, occurring when $q = -2$.

In these two cases, a pair of eigenvalues meets and then breaks off onto the real line yielding an eigenvalue with modulus greater than one and an eigenvalue with modulus less than one. As $q < -2$, we have a pair of reciprocal real positive eigenvalues i.e. λ and $1/\lambda$ and one of them greater than positive one. As q goes to -2 from the left, the pair of reciprocal real eigenvalues

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approach each other at 1. The characteristic multipliers move continuously on the unit circle from $(1,0)$ to $(-1,0)$ as q increases from -2 to 2. As $q > 2$, we have a pair of reciprocal real negative eigenvalues i.e. λ and $1/\lambda$ and one of them less than negative one. As q goes to 2 from the right, the pair of reciprocal real eigenvalues approach each other at -1.

To have linear stability, we need the roots of $p(\lambda)$ on the unit circle, i.e. $q^2 \leq 4$. We begin by analyzing the behavior of the multipliers for the circular case $e = 0$. In this case, the matrix in system (4.17) is constant and therefore, the multipliers can be explicitly computed. The characteristic polynomial of the coefficient matrix of (4.17) is given by

$$\rho^2 - g(a,b,0,\theta)f(a,b,0,\theta),$$

where g, f only depend on the parameters a, b as $e = 0$. If ρ is a root of this polynomial, then $e^{2\pi\rho}$ is a characteristic multiplier. In order to have stability, the root ρ must be purely imaginary. It requires that $h(a,b) = g(a,b,0,\theta)f(a,b,0,\theta)$ is negative.

Using Maple, we have $h(a,b) > 0$ in all possible a, b values as shown in figure 18.

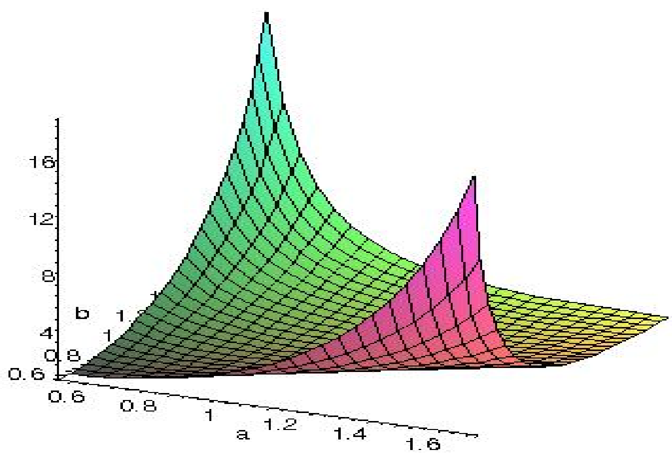

Figure 18: $h(a,b) > 0$ region

Because $h(a,b) = h(b,a)$, $a/\sqrt{3} < b < a\sqrt{3}$, we also can determine the sign of $h(a,b)$ by fixing $a = 1$ as shown in figure 19.

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The minimum of $h(1,b)$ is 0.7836116249 while b=1. This proves that the Kepler orbit for the circular case is unstable.

We now investigate the linear stability of the periodic orbits which are truly elliptic ($e \neq 0$). Since the stability is determined by $q = -tr(M)$, the trace of monodromy matrix, we do not need to calculate the eigenvalues explicitly. If $-2 \leq tr(M) \leq 2$, then the periodic solution is linearly stable. If $|tr(M)| > 2$, it is unstable. Writing a Maple program to calculate the trace of monodromy matrix M, we find all the traces are significantly greater than 2. For example, when a=1, b=1, e varying from 0 to 1 by 1/20, the corresponding traces are

260.3442854,	261.0797976,	263.3111371,	267.1146285,	272.6248514,
280.0463890,	289.6736785,	301.9214558,	317.3731802,	336.8583635,
361.5791520,	393.3279113,	434.8785301,	490.7344415,	568.6778938,
683.3152296,	865.3720827,	1191.302564,	1911.784759,	4478.934516

Table 1: The Traces for a=1, b=1

When $a = 1, b = 1/\sqrt{3} + 1/100$, e varying from 0 to 1 by 1/20, the corresponding traces are

Thus, the periodic solution of the rhombus four body problem is unstable.

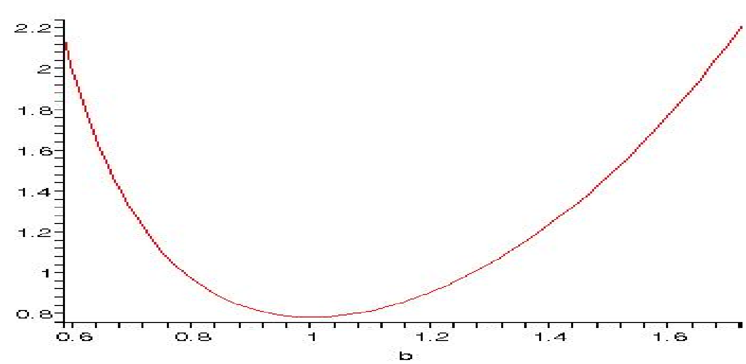

Figure 19: $h(a,b) > 0$ region with a=1

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Remark: We apply this method to study the linear stability of Kepler orbits for the regular polygon N-body problem with one body in the center of the polygon. Because for N equal masses m at the vertices of regular polygon and arbitrary mass μ at the center, the N+1 bodies form a central configuration. We use the symmetry to constrain our solution on regular polygon and get a similar reduced Hamilton system. The corresponding Kepler orbit is unstable if $\mu \leq m$. Although we can not infer that the original Kepler orbit is stable if $\mu > m$, the possible stable Kepler orbit occurs only if $\mu > m$. R. Moeckel [39] studied the linear stability of Kepler orbits with a dominant mass. The regular polygon N-body problem is one of his particular cases.

5 Index Theory for Symplectic Paths and Stability of Periodic Solutions

The main purpose of this section is to show that the index theory for symplectic paths is a very flexible tool in the study of the stability. We will establish the relation between the stability of periodic solution for a Hamiltonian system and its index in low dimensions.

There are infinitely many ways to define index theories for paths of symplectic matrices. A definition of the index theory for symplectic paths is meaningful if it can be applied to different problem. Historically, in the study of closed geodesics on Riemannian Manifolds, M.Morse successfully established his index theory in the 1930s. It was developed by R.Bott [3] in 1956. In terms of the Morse index of the variational problem with periodic or anti-periodic boundary conditions, Daniel Offin gave necessary and sufficient conditions for stability [26] in 2001. Offin also studied hyperbolicity of minimizing geodesics by applying index theory [27].

9684.871521,	9728.151790,	9859.880060,	10085.91046,	10416.67040,
10868.29203,	11464.49530,	12239.64874,	13243.73696,	14550.63459,
16272.39134,	18585.00314,	21777.62330,	26353.37645,	33254.88426,
44429.81158,	64486.29034,	106749.3980,	225015.7447,	861056.0827

Table 2: The Traces for $a = 1, b = 1/\sqrt{3} + 1/100$

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In his book *Index Theory for Symplectic Paths with Applications* [17], Yiming Long introduced an index for symplectic paths in symplectic matrix group. He settled down the main geometric features of this index introducing a characteristic class. Also he built the relations with Morse index and Maslov index. This introductive section is very much guided by this book.

Since the fundamental solution of a general linear hamiltonian system with continuous symmetric periodic coefficients is a path in the symplectic matrix group $Sp(2n)$ starting from the identity, an index theory is established to any symplectic paths. Here the symplectic group is defined by

$$Sp(2n) = \{M \in GL(\mathbb{R}^{2n}) | M^T J M = J\}, \tag{5.1}$$

where

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \tag{5.2}$$

I_n is the identity matrix on $\mathbb{R}^n$, and M^T denotes the transpose of M. For $\tau > 0$, we define the set of symplectic matrix paths by

$$\mathcal{P}_\tau(2n) = \{\gamma \in C([0,\tau], Sp(2n)) | \gamma(0) = I\}. \tag{5.3}$$

Because $Sp(2n)$ is homeomorphic to the product of the unit circle and a simply connected space, a path $\gamma \in \mathcal{P}_\tau(2n)$ rotates naturally in $Sp(2n)$ along this unit circle. The point is to find a way to count this rotation so that the rotation number represents intrinsically the corresponding Morse index of the related Hamiltonian system. For periodic boundary value problems of Hamiltonian systems, because of this consideration, we call a path $\gamma \in \mathcal{P}_\tau(2n)$ *degenerate* if 1 is an eigenvalue of $\gamma(\tau)$, and *non-degenerate* otherwise.

5.1 Floquet Theory and Stability

In this section, we review the Floquet theory and stability theory by following the presentation of Chapter 2 in [5] *Ordinary Differential Equations with Applications* and of Chapter 1 in [11] *Convexity Methods in Hamiltonian Mechanics* .

Consider a system of m linear equations with continuous τ-periodic coefficients:

$$\dot{x} = B(t)x \tag{5.4}$$

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where $B(t)$ is a real $m \times m$ matrix, depending continuously on $t \in \mathbb{R}$ such that:

$$B(t+\tau) = B(t) \tag{5.5}$$

The solutions to the initial value problem:

$$\dot{x} = B(t)x, \qquad x(0) = \xi \in \mathbb{R}^m \tag{5.6}$$

are given by:

$$x(t) = R(t)\xi \tag{5.7}$$

where the matrix $R(t)$ is the principle fundamental matrix solution of system (5.4) with $R(0) = I$. By the general theory of linear systems, the matrix $R(t)$ is invertible for every t, with $R(t)^{-1} = R(-t)$. In the case of a system with periodic coefficients, such as (5.4), Floquet theory gives us some more information.

Indeed, note that if $R(t)$ is the principle fundamental matrix solution of system (5.4), then $R_\tau(t) := R(t+\tau)$ solves problem:

$$\frac{d}{dt}R_\tau(t) = B(t)R_\tau(t), \qquad R_\tau(0) = R(\tau) \tag{5.8}$$

so that $R_\tau(t) = R(t+\tau) = R(t)R(\tau)$. Since $R(\tau)$ is invertible, the set of eigenvalues $\sigma(R(\tau))$ does not contain 0. Choose a simply connected domain Ω, and a determination of the logarithm, $\log : \Omega \mapsto \mathbb{C}$, such that $\sigma(R(\tau))$ is contained in Ω. By standard results on Banach algebras, $\log A$ is well-defined for all matrices A whose spectrum is contained in Ω, and it is a holomorphic function of A. In fact, we have the formula:

$$f(A) = (2i\pi)^{-1} \int (zI - A)^{-1} \log z dz \tag{5.9}$$

where the integral is taken over any closed curve in Ω winding once around $\sigma(A)$. Define a matrix C with complex coefficients by $C := \tau^{-1} \log R(\tau)$. We have

$$R(\tau) = \exp C\tau \tag{5.10}$$

$$\sigma C = \tau^{-1} \log \sigma(R(\tau)) \tag{5.11}$$

$$C \text{ and } R(\tau) \text{ have the same invariant subspaces.} \tag{5.12}$$

We now understand system (5.4) as in a differential equation in $\mathbb{C}^m$. We have

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Theorem 5.1. *(Floquet's Theorem). If $R(\tau)$ is a principle fundamental matrix solution of the τ-periodic system (5.4), then, for all $t \in \mathbb{R}$,*

$$R(t+\tau) = R(t)R(\tau) \tag{5.13}$$

In addition, for each possibly complex matrix C such that

$$e^{\tau C} = R(\tau), \tag{5.14}$$

there is a possibly complex τ-periodic matrix function $t \mapsto P(t)$ such that $R(t) = P(t)e^{tC}$ for all $t \in \mathbb{R}$. Also, there is a real matrix S and a real $2T$-periodic matrix function $t \mapsto Q(t)$ such that $R(t) = Q(t)e^{tS}$ for all $t \in \mathbb{R}$.

The representation $R(t) = P(t)e^{tC}$ in Floquet's theorem 5.1 is called a *Floquet normal form* for the fundamental matrix $R(t)$. We will use this normal form to study the stability of the zero solution of periodic linear system (5.4).

The eigenvalues of $R(\tau)$ are called the *Floquet multipliers*: they are uniquely defined. The eigenvalues of C are called the *Floquet exponents*: they depend on the particular choice of C, which is related to $R(\tau)$ by the equation $R(\tau) = \exp(C\tau)$. If we denote by λ_i the Floquet multipliers, and by ω_i the Floquet exponents, properly ordered, we have the obvious relation:

$$\lambda_i = \exp(\omega_i) \text{ for } 1 \leq i \leq m$$

so that the Floquet multipliers give the Floquet exponents modulo $2\pi i\tau^{-1}$. Note for instance that system (5.4) has a periodic solution if and only if 1 is a Floquet multiplier. More generally, the system (5.4) has a $k\tau$-periodic solution if and only if one of the Floquet multipliers is a k-th root of unity.

We now turn to question of stability. The system (5.4) is called *positively (resp. negatively) stable* if all its real solutions remain bounded for all $t > 0$ (resp. $t < 0$). It is called *stable* if it is both positively and negatively stable, that is, its real solutions are bounded for all times $t \in \mathbb{R}$.

Denote by $\mathfrak{D}$ the unit disk in $\mathbb{C}$, and by U the unit circle:

$$\mathfrak{D} := \{z : |z| \leq 1\} \tag{5.15}$$

$$U := \{z : |z| = 1\} \tag{5.16}$$

Lemma 5.1. The system (5.4) is positively stable if and only if $R(\tau)$ is diagonalizable and its spectrum lies entirely in $\mathfrak{D}$. It is stable if and only if it is diagonalizable and its spectrum lies

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entirely on U.

The question of stability becomes more delicate if additional restrictions are put on the system, namely, that it must be Hamiltonian.

Definition 5.1. The linear system (5.4) is called Hamiltonian if its dimension is even, $m = 2n$, and we have $B(t) = JA(t)$, where $A(t)$ is a symmetric matrix and the matrix J is given by equation (5.2).

Form now on, we shall consider Hamiltonian systems only. We are given a matrix $A(t)$, symmetric, $\tau-$periodic, depending continuously on t:

$$A(t) = A^T(t), \qquad\qquad A(t+\tau) = A(t),$$

and we are interested in the linear differential system:

$$\dot{x} = JA(t)x. \tag{5.17}$$

The fundamental property of such a system is that its principle fundamental matrix solution $\gamma(t)$ is symplectic, that is, $\gamma(t) \in \mathcal{P}_\tau(2n)$ which is given by equation (5.3). Then we have the following properties.

Lemma 5.2. $\gamma(t)$ preserves volume and orientation:

$$Det\gamma(t) = 1.$$

Lemma 5.3. If λ is a Floquet multiplier, so are its inverse λ^{-1}, its complex conjugate $\bar{\lambda}$, and $\bar{\lambda}^{-1}$. They all have the same multiplicity. If 1 or -1 is a Floquet multiplier, it must have even multiplicity.

Definition 5.2. The system (5.17) is called *stable* if all its solutions remain bounded when $t \in \mathbb{R}$. It is *strongly stable* if there exists some $\epsilon > 0$ such that, whenever $B(t)$ is a symmetric τ-periodic matrix, depending continuously on t, with

$$\|B(t) - A(t)\| \leq \epsilon, \qquad \forall t$$

the system $\dot{x} = JB(t)x$ is stable.

Definition 5.3. A symplectic matrix M is called *stable* if all its iterations M^k remain bounded when $k \in \mathbb{Z}$. It is called *strongly stable* if there exists some $\epsilon > 0$ such that all symplectic matrices N with $\|M - N\| \leq \epsilon$ are stable.

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Lemma 5.4. System (5.17) is strongly stable if and only if the symplectic matrix $\gamma(\tau)$ is strongly stable.

From lemma 5.1, system (5.17) is stable if and only if the symplectic matrix $\gamma(\tau)$ is diagonalizable and its spectrum lies entirely on U.

Let $H \in C^2(\mathbb{R}/(\tau\mathbb{Z}) \times \mathbb{R}^{2n}, \mathbb{R})$. Suppose x is a $\tau-$periodic solution of the Hamiltonian system

$$\dot{x}(t) = JH'(t, x(t)), \tag{5.18}$$

such that H is C^2 along the orbit $x(\mathbb{R})$ of x. Its linearization along the orbit $x(t)$ is

$$\dot{\xi} = JH''(x(t))\xi \tag{5.19}$$

which is a linear Hamiltonian system. The associated symplectic path of x is defined to be the principle fundamental solution γ_x of the linearized Hamiltonian system (5.19). The eigenvalues of $\gamma_x(\tau)$ are called Floquet multipliers of the solution x of the Hamiltonian system. By Floquet theorem, the periodic solution is linearly stable if all the Floquet multipliers are on unit circle and $\gamma_x(\tau)$ is diagonalizable.

5.2 Index Theory for Symplectic Paths

We are going to study the global structure of the symplectic group $Sp(2n)$ and build the relationship between the linear stability of Hamiltonian system (5.18) and index theory.

5.2.1 Definition of Index Theory

In this subsection, we will use the same notation and follow the presentation of chapter 2 and chapter 5 in Y. Long's book *Index Theory for Symplectic Paths with Application* [17] to define an index for Symplecitic paths.

Let U be the unit circle in the complex plane C. For any $\omega \in U$ and $M \in Sp(2n)$, we define

$$D_\omega(M) = (-1)^{n-1}\omega^{-n}\det(M - \omega I).$$

Then D is a real smooth function on $U \times Sp(2n)$. According to the value of D_ω, we define some subset of $Sp(2n)$. For $\omega \in U$, we define the $\omega-$singular set $Sp(2n)^0_\omega$ of $Sp(2n)$ and its subsets

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$\mathcal{M}_\omega^k(2n)$ with $0 \leq k \leq 2n$ by

$$Sp(2n)_\omega^0 = \{M \in Sp(2n) | D_\omega(M) = 0\},$$

$$\mathcal{M}_\omega^k(2n) = \{M \in Sp(2n) | \nu_\omega(M) = k\}$$

where $\nu_\omega(M) = dim_C ker_C(M - \omega I)$. We also define the $\omega-$regular sets of $Sp(2n)$ by

$$Sp(2n)_\omega^\pm = \{M \in Sp(2n) | \pm D_\omega(M) < 0\}$$

$$Sp(2n)_\omega^* = Sp(2n)_\omega^+ \cup Sp(2n)_\omega^-$$

For $\tau > 0$ and $\omega \in U$, we further define the set of $\omega-$non-degenerate paths by

$$\mathcal{P}_{\tau,\omega}^*(2n) = \{\gamma \in \mathcal{P}_\tau(2n) | \gamma(\tau) \in Sp(2n)_\omega^*\},$$

and the set of $\omega-$degenerate paths by

$$\mathcal{P}_{\tau,\omega}^0(2n) = \mathcal{P}_\tau(2n) \backslash \mathcal{P}_{\tau,\omega}^*(2n)$$

For any symplectic matrix $M \in Sp(2n)$, it can be represented in the form (polar decomposition)

$$M = AU,$$

where $A = \sqrt{MM^T}$ is a symmetric symplectic positive definite matrix, U is a symplectic orthogonal matrix, and they are uniquely determined by M. By the polar decomposition, it is easy to prove that the symplectic group $Sp(2n)$ is homeomorphic to the topological product of the unit circle U in the complex plane C and a simply connected topological Space.

For $n = 1$, Y. Long in [17] introduce a geometric representation of $Sp(2)$ in $\mathbb{R}^3$. For any matrix $M \in Sp(2)$, by the polar decomposition, M can be written in the form

$$M = \begin{pmatrix} r & z \\ z & (1+z^2)/r \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{5.20}$$

where $(r, \theta, z) \in \mathbb{R}^+ \times S^1 \times \mathbb{R}$, $\mathbb{R}^+ = \{r \in \mathbb{R} | r > 0\}$, $S^1 = \mathbb{R}/(2\pi\mathbb{R} - \pi)$, and (r, θ, z) is uniquely determined by M. Viewing (r, θ, z) as the cylindrical coordinates in $\mathbb{R}^3$, we obtain a representation of $Sp(2)$ in $\mathbb{R}^3$. Under this representation, it is easy to see that the two eigenvalues of M are

$$\lambda = \frac{1}{2r}((r^2 + z^2 + 1)\cos\theta \pm \sqrt{(1 + r^2 + z^2)^2 \cos^2\theta - 4r^2}),$$

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which are either two reciprocal real numbers or two conjugate complex numbers on the unit circle U in the complex plane C. For $\omega = \cos\varphi + \sqrt{-1}\sin\varphi \in U$ and M in the form (5.20), we obtain

$$D_\omega(M) = 2\cos\varphi - \left(r + \frac{1+z^2}{r}\right)\cos\theta.$$

Then we have

$$Sp(2)^{\pm}_{\omega} = \{(r,\theta,z) \in \mathbb{R}^+ \times S^1 \times \mathbb{R} | \pm (r^2+z^2+1)\cos\theta > 2r\cos\varphi\},$$

$$Sp(2)^{0}_{\omega} = \{(r,\theta,z) \in \mathbb{R}^+ \times S^1 \times \mathbb{R} | \pm (r^2+z^2+1)\cos\theta = 2r\cos\varphi\}.$$

Let $Sp(2)^0_{\omega,\pm} = \{(r,\theta,z) \in Sp(2)^0_\omega | \pm\sin\theta > 0\}$. Then we have $Sp(2)^0_{\pm 1} = Sp(2)^0_{\pm 1,+} \bigcup \{\pm I\} \bigcup Sp(2)^0_{\pm 1,-}$, and $Sp(2)^0_\omega = Sp(2)^0_{\omega,+} \bigcup Sp(2)^0_{\omega,-}$ for $\omega \in U \backslash \mathbb{R}$. Here we are especially interested in the cylindrical coordinate representation of the singular hypersurfaces $Sp(2)^0_\omega$ for $\omega \in U$.

In figure 20 (see [17], p.59), this $\mathbb{R}^3-$cylindrical coordinate representation of $Sp(2)^0_1$ is given with the Descartes coordinates $(x,y,z) = (r\cos\theta, r\sin\theta, z)$.
In figure 21, the intersection of the plane $\{z = 0\}$ with $Sp(2)^0_1, Sp(2)^0_{-1}$, and $Sp(2)^0_\omega$ for some $\omega \in U$ are given. Lemma 5.5, lemma 5.6 and lemma 5.7 are easily derived from [17]

Lemma 5.5. For any $\omega \in U$ the set $Sp(2)^*_\omega$ possesses precisely two path connected components $Sp(2)^+_\omega$ and $Sp(2)^-_\omega$, and it holds that $D(2) = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \in Sp(2)^+_\omega$ and $D(-2) =$

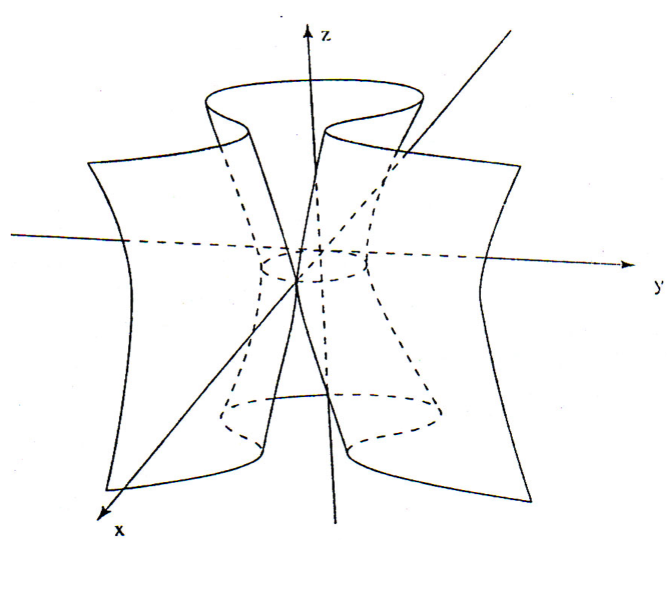

Figure 20: The $\mathbb{R}^3$-cylindrical coordinate representation of $Sp(2)^0_1$

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$$\begin{pmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \in Sp(2)_\omega^-.$$

Lemma 5.6. Fix an $\omega \in U\backslash\mathbb{R}$. $Sp(2)_1^-, Sp(2)_{-1}^+$ and $Sp(2)_\omega$ are homeomorphic to $\mathbb{R}^3$. $Sp(2)_1^+$ and $Sp(2)_{-1}^-$ are homeomorphic to $\mathbb{R}^3\backslash\{(x,y,z) \in \mathbb{R}^3 | x^2+z^2 \le y^2\}$.
Thus for any $\omega \in U$, the set $Sp(2)_\omega^*$ is simply connected in $Sp(2)$, i.e., any closed curve inside $Sp(2)_\omega^+$ or $Sp(2)_\omega^-$ can be continuously contracted inside $Sp(2)$ to a point.

Lemma 5.7. When $\omega = 1$, in the singular hypersurface $Sp(2)_1^0$, the identity matrix I_2 is the only element which satisfies $\nu_1(I_2) = 2$. The regular part $\mathcal{M}(2)$ of $Sp(2)_1^0$ possesses precisely two path connected components $Sp(2)_{1,+}^0$ and $Sp(2)_{1,-}^0$, both of which are smooth hypersurfaces diffeomorphic to $\mathbb{R}^2\backslash\{0\}$.

Note that from the $\mathbb{R}^3-$cylindrical coordinate represntation introduced above, $Sp(2)_\omega^0$ for $\omega \in U$ is orientable. Now we can define the index for symplectic paths in $Sp(2)$ and index for periodic solutions of Hamiltonian system.

Fix $\tau > 0$, $\omega \in U$, we use the concept of intersection numbers in the algebraic topology to

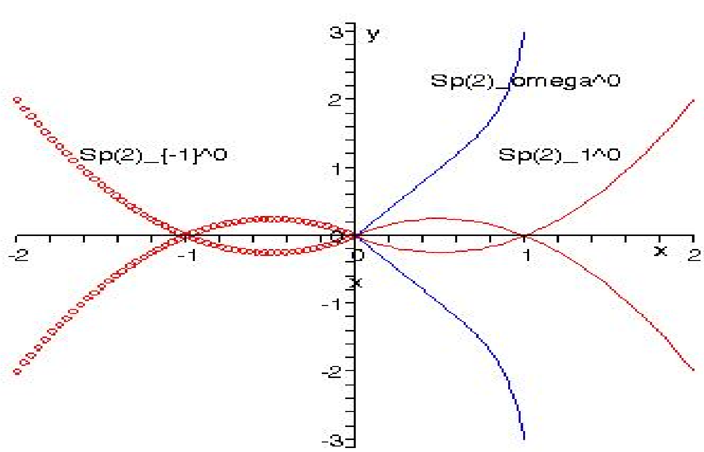

Figure 21: Singular Sets: $Sp(2)_1^0$ and $Sp(2)_\omega^0$

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define index for $\omega-$non-degenerate paths in $\mathcal{P}^*_{\tau,\omega}$. Because an orientation of $Sp(2)^0_\omega$ is defined, $Sp(2)^0_\omega$ form a locally finite $2-$dimensional singular homological cycle [29]. For any $\tau > 0$, let $\xi_+(t) = \begin{pmatrix} 1+\frac{t}{\tau} & 0 \\ 0 & \frac{\tau}{t+\tau} \end{pmatrix}$ for all $t \in [0,\tau]$. For any $\tau > 0$ and a path $\beta \in \mathcal{P}_\tau(2)$, we denote $\beta^{-1}(t) = \beta(\tau - t)$ for $t \in [0,\tau]$.

Definition 5.4. For any $\omega \in U$, $\tau > 0$, and $\gamma \in \mathcal{P}^*_{\tau,\omega}(2)$, we define

$$i_\omega(\gamma) = [Sp(2)^0_\omega : \gamma * \xi_+^{-1}]. \tag{5.21}$$

For any path $\gamma \in \mathcal{P}^*_{\tau,\omega}(2)$, the two end points of the joint path $\gamma * \xi_+^{-1}$ are not located on $Sp(2)^0_\omega$. Thus the algebraic homological intersection number in (5.21) is well defined. The integer $i_\omega(\gamma)$ is called the index of the symplectic path γ.

To further explain this definition, fixing an $\omega \in U$, we consider the smooth paths first. Let $\varphi \in C^1([0,\tau], Sp(2))$ such that $\varphi(0) = D(2)$ and $\varphi(\tau) \in Sp(2)^*_\omega$. Then the direction of φ at the point $\varphi(t)$ is defined to be the tangent direction $\dot{\varphi}(t)$ of φ at that point. Now we assume the following conditions on φ.

(1) It holds that

$$\varphi([0,\tau]) \bigcap Sp(2)^0_\omega \subset \varphi([0,\tau]) \bigcap \mathcal{M}^1_\omega(2) \equiv S(\omega,\varphi).$$

(2) φ intersects $\mathcal{M}^1_\omega(2)$ transversally, i.e. at any intersection point $\varphi(t) \in S(\omega,\varphi)$, the tangent vector of φ at the point $\varphi(t)$ is not contained in the tangent plane of $\mathcal{M}^1_\omega(2)$ at the same point, i.e. $\dot{\varphi}(t) \cdot \eta(\omega, \varphi(t)) \neq 0$,, where $\eta(\omega, x)$ is the positively directed unit normal vector of $\mathcal{M}^1_\omega(2)$ at its point x.

Denote by $\mathcal{C}^1_{\tau,reg}(2)$ the set of all such C^1 curves satisfying (1) and (2), and call them the regular curves in $Sp(2)$. Under these tow conditions, we define the intersection number $\mu(\varphi, \mathcal{M}^1(2), x)$ at $\varphi(t) \in S(\omega,\varphi)$ by

$$\mu(\varphi, \mathcal{M}, \varphi(t)) = 1, \text{ if } \dot{\varphi}(t) \cdot \eta(\omega, \varphi(t)) > 0,$$

$$\mu(\varphi, \mathcal{M}, \varphi(t)) = -1, \text{ if } \dot{\varphi}(t) \cdot \eta(\omega, \varphi(t)) < 0.$$

Then the intersection number φ and $Sp(2)^0_\omega$ is defined by

$$[Sp(2)^0_\omega : \varphi] = \sum_{x \in S(\omega,\varphi)} \mu(\varphi, \mathcal{M}, x), \ \forall \varphi \in \mathcal{C}^1_{\tau,reg}(2).$$

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Denote by $\mathcal{C}_\tau^1(2)$ the set of paths in $C^1([0,\tau], Sp(2))$ started from $D(2)$. Let $\mathcal{C}^1_{\tau,reg}(2, D(2)) = \mathcal{C}_\tau^1(2) \bigcap \mathcal{C}^1_{\tau,reg}(2)$. Now fix a path $\gamma \in \mathcal{P}_\tau(2)$. we consider the C^0- approximations $\varphi \in \mathcal{C}^1_{\tau,reg}(2, D(2))$ of $\gamma * \xi_+^{-1}$ satisfying $\varphi(\tau) = \gamma(\tau)$. It is easy to see that if φ is sufficiently C^0-close to $\gamma * \xi_+^{-1}$, the intersection umber $[Sp(2)^0_\omega : \varphi]$ is independent of the particular choices of φ. Therefore we obtain

$$[Sp(2)^0_\omega : \gamma * \xi_+^{-1}] = [Sp(2)^0_\omega : \varphi]$$

for all such smooth C^0-approximations of $\gamma * \xi_+^{-1}$. Thus definition 5.4 is well defined.

For $\tau > 0$ and $\omega \in U$, given two paths γ_0 and $\gamma_1 \in \mathcal{P}_\tau(2n)$, if there exists a map $\delta \in C([0,1] \times [0,\tau], Sp(2n))$ such that $\delta(0,\cdot) = \gamma_0(\cdot), \delta(1,\cdot) = \gamma_1(\cdot)$, $\delta(s,0) = I$, and $\nu_\omega(\delta(s,\cdot))$ is constant for $0 \le s \le 1$, then γ_0 and γ_1 are $\omega-$homotopic on $[0,\tau]$ along $\delta(\cdot,\tau)$ and we write $\gamma_0 \sim_\omega \gamma_1$.

Note that the index function i_ω is homotopy invariant, i.e., for any $\omega \in U$, γ_0 and $\gamma_1 \in \mathcal{P}_\tau(2)$, $\gamma_0 \sim_\omega \gamma_1$ implies $i_\omega(\gamma_0) = i_\omega(\gamma_1)$. Conversely it is also true. In figure 22, γ is $1-$homotopic to α_1 on $[0,\tau]$ which is defined in the proof of theorem 5.2 but γ is not $-1-$homotopic to α_1 on $[0,\tau]$. In fact, $i_1(\gamma) = 1$ but $i_{-1}(\gamma) = 0$.

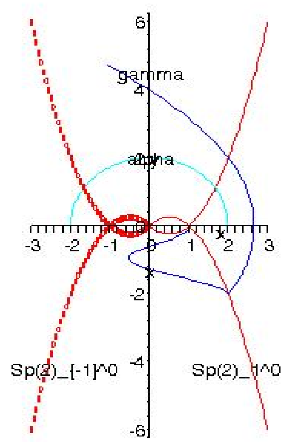

Figure 22: 1-homotopy class of α_1

For an $\omega-$degenerate path, i.e., fixed $\tau > 0$, and $\omega \in U$, $\forall \gamma \in \mathcal{P}^0_{\tau,\omega}(2)$, there are two methods to define the index function $i_\omega(\gamma)$. One is perturbation method and another is minimizing method.

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More details can be found in Chapter 5.1 in [17]. So for any path $\gamma \in \mathcal{P}_\tau(2)$, an index function $i_\omega(\gamma)$ is well defined for all $\omega \in U$.

5.2.2 Some Properties of Index Functions

Base on the above definition and lemmas, we prove the folloowing theorems which build a relation between index function for a syplectic path in $Sp(2)$ and the ending point of the path.

Theorem 5.2. *For $\tau > 0$, $\forall \gamma \in \mathcal{P}^*_{\tau,1}(2)$, we have:*

(1) $i_1(\gamma)$ is an even integer if and only if $\gamma(\tau) \in Sp(2)^+_1$. Furthermore, if λ_1, λ_2 are two eigenvalues of $\gamma(\tau)$, then $0 < \lambda_1 < 1 < \lambda_2$ and $\lambda_1 = \frac{1}{\lambda_2}$.

(2) $i_1(\gamma)$ is an odd integer if and only if $\gamma(\tau) \in Sp(2)^-_1$.

Proof. For $\tau > 0$, $\gamma \in \mathcal{P}^*_{\tau,1}(2)$, assume $i_1(\gamma) = k$, we define a *zigzag standard path* α_k in $\mathcal{P}^*_{\tau,1}(2)$ such that $\alpha_k \sim_1 \gamma$ as follows. Set

$$\phi_{\tau,\theta}(t) = \begin{pmatrix} \cos(\theta\frac{t}{\tau}) & -\sin(\theta\frac{t}{\tau}) \\ \sin(\theta\frac{t}{\tau}) & \cos(\theta\frac{t}{\tau}) \end{pmatrix}, \forall t \in [0,\tau], \theta \in \mathbb{R},$$

$$\alpha_0(t) = \xi_+(t), \forall t \in [0,\tau],$$

For $0 \le t \le \tau$ we define

$$\alpha_k(t) = [D(2)\phi_{\tau,k\pi}] * \xi_+(t), \text{ if } k \in \mathbb{Z}\backslash\{0\}.$$

Then $\alpha_k(\tau) = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ if k is even and $\alpha_k(\tau) = \begin{pmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$ if k is odd. $\alpha_k \in \mathcal{P}^*_{\tau,1}(2)$ and $i_1(\alpha_k) = k$.

Claim: Both $\gamma(\tau)$ and $\alpha_k(\tau)$ are in the same 1-regular subset $Sp(2)^+_1$ or in $Sp(2)^-_1$.

Because $i_1(\gamma) = k = i_1(\alpha_k)$, $\gamma \sim_1 \alpha_k$. There exists a map $\delta \in C([0,1] \times [0,\tau], Sp(2))$ such that $\delta(0,\cdot) = \gamma(\cdot)$, $\delta(1,\cdot) = \alpha_k(\cdot)$, $\delta(s,0) = I$, and $\nu_1(\delta(s,\cdot)) = \nu_1(\delta(s,\tau)) = dim_C ker_C(\delta(s,\tau) - I) = 0$ for $0 \le s \le 1$. By the continuity of $\delta(\cdot,\tau)$, both $\gamma(\tau)$ and $\alpha_k(\tau)$ must be in the same 1-regular subset $Sp(2)^+_1$ or in $Sp(2)^-_1$.

Then if $i_1(\gamma) = k$ is even, $\alpha_k(\tau) \in Sp(2)^+_1$. We complete the proof of the first part of (1). As for

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the second part of (1), it is directed from the properties of eigenvalues of sypmlectic matrix. The proof for (2) is similar as the proof for (1). ♮

Theorem 5.3. *For $\tau > 0$, $\forall \gamma \in \mathcal{P}^*_{\tau,-1}(2)$, we have:*

(1) $i_{-1}(\gamma)$ is an even integer if and only if $\gamma(\tau) \in Sp(2)^+_{-1}$.

(2) $i_{-1}(\gamma)$ is an odd integer if and only if $\gamma(\tau) \in Sp(2)^-_{-1}$.Furthermore, if λ_1, λ_2 are two eigenvalues of $\gamma(\tau)$, then $0 > \lambda_1 > -1 > \lambda_2$ and $\lambda_1 = \frac{1}{\lambda_2}$.

We will omit the proof of theorem 5.3 because it is similar to the proof of theorem 5.2. From theorem 5.2 and 5.3, we obtain

Corollary 5.1. For $\tau > 0$, $\forall \gamma \in \mathcal{P}^*_{\tau,\pm 1}(2)$, $i_1(\gamma)$ is an odd integer and $i_{-1}(\gamma)$ is an even integer if and only if all eigenvalues of $\gamma(\tau)$ are on the unit circle U.

5.3 Index for periodic solutions of Hamiltonian system and its stability zone

In this subsection, first we define the index for linear Hamiltonian system. Then we establish the stability zone according to the properties of index for the linear Hamiltonian system. Theorem 5.2 and theorem 5.3 will play important roles.

Let $x(t)$ be a periodic solution of Hamiltonian system (5.18) and $\gamma_x(t)$ be principle fundamental matrix solution of its associated linear Hamiltonian system (5.19). By Floquet theorem, the periodic solution $x(t)$ is linearly stable if and only if all the Floquet multipliers are on unit circle and $\gamma_x(\tau)$ is diagonalizable.

For $\omega \in U$, we define the index function of x via that of its associated symplectic path γ_x : $i_\omega(x) = i_\omega(\gamma_x)$.

Theorem 5.4. *Let the Hamiltonian system (5.18) have 1 degree of freedom, i.e. $n = 1$. Suppose x is a $\tau-$periodic solution of the Hamiltonian system (5.18) and γ_x is the associated symplectic*

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path of x. If $\gamma_x \in \mathcal{P}_{\tau,1}(2)^*$ and $i_1(\gamma_x)$ is an even integer, then the periodic solution x is linear unstable.

Proof. Because the index $i_1(\gamma_x)$ of the periodic solution is an even integer, $\gamma_x(\tau) \in Sp(2)_1^+$. So the Floquet multipliers are all real and one is bigger than 1. Therefore the periodic solution is linear unstable.

Theorem 5.5. *Let the Hamiltonian system (5.18) be of freedom 1, i.e. $n = 1$. Suppose x is a $\tau-$periodic solution of the Hamiltonian system (5.18) and γ_x is the associated symplectic path of x. If $\gamma_x \in \mathcal{P}_{\tau,-1}(2)^*$ and $i_{-1}(\gamma_x)$ is an odd integer, then the periodic solution x is linear unstable.*

Corollary 5.2. Suppose x is a $\tau-$periodic solution of the Hamiltonian system (5.18) with $n = 1$ and γ_x is the associated symplectic path of x. If $\gamma_x \in \mathcal{P}_{\tau,\pm 1}(2)^*$ and the periodic solution x is linearly stable, then $i_1(\gamma_x)$ is an odd integer and $i_{-1}(\gamma_x)$ is an even integer.

Our final result shows that these two theorems are in fact sufficient for stability as well. Suppose x is a $\tau-$periodic solution of the Hamiltonian system (5.18) and γ_x is the associated symplectic path of x. If $\gamma_x \in \mathcal{P}_{\tau,\pm 1}(2)^*$, we denote the index of the periodic solution x by $ind(x) = i_1(\gamma_x)+i_{-1}(\gamma_x)$.

Theorem 5.6. *Let the Hamiltonian system (5.18) be of freedom 1, i.e. $n = 1$. Suppose x is a $\tau-$periodic solution of the Hamiltonian system (5.18) and γ_x is the associated symplectic path of x. If $\gamma_x \in \mathcal{P}_{\tau,\pm 1}(2)^*$ and $ind(x)$ is odd if and only if the periodic solution x of the Hamiltonian system (5.18) is linearly stable. Or equivalently, $ind(x)$ is even if and only if the periodic solution x of the Hamiltonian system (5.18) is linear unstable.*

Proof. $Ind(x)$ is even if and only if $i_1(\gamma_x)$ and $i_{-1}(\gamma_x)$ are both even or odd. If $i_1(\gamma_x)$ and $i_{-1}(\gamma_x)$ are both even, by theorem 5.4, $i_1(\gamma_x)$ even implies that the periodic solution x is linear unstable. If $i_1(\gamma_x)$ and $i_{-1}(\gamma_x)$ are both odd, by theorem 5.5, $i_{-1}(\gamma_x)$ odd implies that the periodic solution x is linear unstable.
Conversely, assuming x is linear unstable, if $i_1(\gamma_x)$ is an even integer, then $\gamma_x(\tau) \in Sp(2)_1^+$

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by theorem 5.2. But if $i_{-1}(\gamma_x)$ is an odd integer, then $\gamma_x(\tau) \in Sp(2)^-_{-1}$ by theorem 5.3. Because $Sp(2)^+_1 \bigcap Sp(2)^-_{-1}$ is empty, there are no such symplectic paths. So $i_{-1}(\gamma_x)$ must be also an even integer. Therefore $ind(x)$ is even. Similar discussion can be apply to other case ($i_1(\gamma_x)$ is an odd integer). $\natural$

For higher dimensional problems, that is when $n > 1$, there is much less that can be stated for linear systems with periodic coefficients in these simple terms. But for $n = 2$, some instability and stability results can be determined by parity of index. Those results can be applied to study the stability of N-body problem such as isosceles three body problem [28],[34] and figure eight solution [8],[41].

5.4 Stability and $SP(4)$

In this section, we first review the definition of index for symplectic paths with $n-$freedom by following the presentation of chapter 5.2 in [17].

For any $M \in Sp(2n)$, it has a unique polar decomposition $M = AU$. We denote by

$$U(M) = \begin{pmatrix} u_1(M) & -u_2(M) \\ u_2(M) & u_1(M) \end{pmatrix}$$

the orthogonal and symplectic part of its unique polar decomposition. Then $u(M) = u_1(M) + \sqrt{-1}u_2(M) \in U(n, \mathbb{C})$. So in such a way, for every path $\gamma \in \mathcal{P}_\tau(2n)$, we can uniquely associate to it a path,

$$u_\gamma(t) = u(\gamma(t)), \qquad \forall t \in [0, \tau]$$

in the unitary group $U(n, \mathbb{C})$. For any $\gamma \in C([0, \tau], Sp(2n))$, let $\triangle : [0, \tau] \to \mathbb{R}$ be any continuous real function satisfying

$$\det u_\gamma(t) = \exp(\sqrt{-1}\triangle(t)), \qquad \forall t \in [0, \tau]. \tag{5.22}$$

We define the rotation number of γ by

$$\triangle_\tau(\gamma) = \triangle(\tau) - \triangle(0).$$

Then $\triangle_\tau(\gamma)$ depends only on γ but not on the choice of the function $\triangle$ satisfying (5.22).

For any $\omega \in U$ and $\gamma \in \mathcal{P}^*_{\tau,\omega}(2n)$, by Theorem 2.4.1 in [17] we can connect $\gamma(\tau)$ to M_n^+ or M_n^- by

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a path $\beta : [0, \tau] \to Sp(2n)^*_\omega$. Under these conditions, define

$$k \equiv \triangle_\tau(\beta * \gamma)/\pi. \tag{5.23}$$

Then k is an integer; is independent of the choice of the path β, and

$$\begin{cases} k & \text{is odd,} \quad \text{if } \beta(\tau) = M_n^-, \\ k & \text{is even,} \quad \text{if } \beta(\tau) = M_n^+, \end{cases} \tag{5.24}$$

We denote by $\mathcal{P}^k_{\tau,\omega}(2n)$ the set of all such paths in $\mathcal{P}^*_{\tau,\omega}(2n)$, that possess the property (5.23).

Definition 5.5. For any $\omega \in U$ and $\tau > 0$, we define the index of a symplectic path γ by

$$i_\omega(\gamma) = k, \qquad \text{if } \gamma \in \mathcal{P}^k_{\tau,\omega}(2n). \tag{5.25}$$

Lemma 5.8. (Lemma 1, P.43 of Y.Long [17] and G. Roberts [40]) For $M \in Sp(4)$, its characteristic polynomial has the form

$$f_M(\lambda) = \lambda^4 - 4A\lambda^3 + B\lambda^2 - 4A\lambda + 1 \tag{5.26}$$

where $A = \frac{\mathrm{tr}(M)}{4}, B = \frac{1}{2}((\mathrm{tr}(M))^2 - \mathrm{tr}(M^2))$. Then

1^o (5.26) possesses one pair of conjugate double roots if and only if $B = 4A^2 + 2$.

2^o $+1$ is one root of (5.26) if and only if $B = 8A - 2$.

3^o -1 is one root of (5.26) if and only if $B = -8A - 2$.

4^o $i = \sqrt{-1}$ (or $-i = -\sqrt{-1}$) is one root of (5.26) if and only if $B = 2$.

Proof. By direct computation. $\natural$

Figure 23 in AB-plane is obtained as follows (A similar graph in [40] is used to analyze the linear stability of equilateral triangle solution in 3-body problem).

$\langle 1\rangle$ Let R be the tangent point of the line $B = -8A - 2$ to the curve $B = 4A^2 + 2$. Let S be the tangent point of the line $B = 8A - 2$ to the curve $B = 4A^2 + 2$. Denote by T the intersection point of the line $B = -8A - 2$ and the line $B = 8A - 2$. It is easy to check that $R = (-1, 6), S = (1, 6), T = (0, -2)$.

$\langle 2\rangle$ We denote by I the open region bounded by the curve $B = 4A^2 + 2$ from left to R and the line $B = -8A - 2$ from left to R. Other open regions are illustrated as in Figure 23.

$\langle 3\rangle$ Note that for any $M \in Sp(4)$, there corresponds a unique point (A, B) in the $AB-$plane, where A, B are the parameters in the characteristic polynomial of M in (5.26). If (A, B) is in region I,

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we may say the symplectic matrix M is in region I and similarly for other regions.

$\langle 4 \rangle$ Let $\sigma(M) = \{\lambda | f_M(\lambda) = 0\}$ be the set of eigenvalues of M. By direct computation, we have:

(1) $\sigma(M) \subseteq \{\lambda_1, \lambda_2, \lambda_1^{-1}, \lambda_2^{-1} | \lambda_1 < \lambda_2 < -1\}$ if M is in the open region I.

(2) $\sigma(M) \subseteq \{\lambda_1, \bar{\lambda}_1, \lambda_1^{-1}, \bar{\lambda}_1^{-1} | \lambda_1 = a + b\sqrt{-1}, b \neq 0, \|\lambda_1\| \neq 1\}$ if M is in the open region II.

(3) $\sigma(M) \subseteq \{\lambda_1, \lambda_2, \lambda_1^{-1}, \lambda_2^{-1} | 1 < \lambda_1 < \lambda_2\}$ if M is in the open region III.

(4) $\sigma(M) \subseteq \{\lambda_1, \lambda_2, \lambda_1^{-1}, \lambda_2^{-1} | 1 < \lambda_1, \lambda_2 = a + b\sqrt{-1}, b \neq 0, \|\lambda_2\| = 1\}$ if M is in the open region IV.

(5) $\sigma(M) \subseteq \{\lambda_1, \lambda_2, \lambda_1^{-1}, \lambda_2^{-1} | \lambda_1 < -1, 1 < \lambda_2\}$ if M is in the open region V.

(6) $\sigma(M) \subseteq \{\lambda_1, \lambda_2, \lambda_1^{-1}, \bar{\lambda}_2 | \lambda_1 < -1, \lambda_2 = a + b\sqrt{-1}, b \neq 0, \|\lambda_2\| = 1\}$ if M is in the open region VI.

(7) $\sigma(M) \subseteq \{\lambda_1, \lambda_2, \bar{\lambda}_1, \bar{\lambda}_2 | \lambda_i = a_i + b_i\sqrt{-1}, b_i \neq 0, \|\lambda_i\| = 1, \lambda_1 \neq \lambda_2, \lambda_1 \neq \bar{\lambda}_2, i = 1, 2\}$ if M is in the open region VII.

Recall for any $\omega \in U, M \in Sp(2n)$, the $\omega-$nullity of M is $\nu_\omega(M) = \dim_C \ker_C(M - \omega I)$ and $\sigma(M)$ is the set of eigenvalues of M. We define the *hyperbolic index* $\alpha(M)$ of M be the $\mod 2$ number of the total algebraic multiplicity of negative eigenvalues of M which are strictly less than

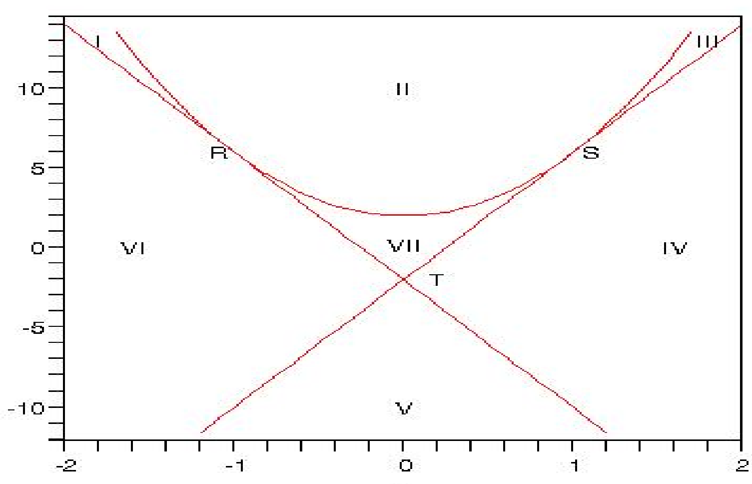

Figure 23: Stable Region on AB-plane

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-1, and the *elliptic height* $e(M)$ of M by the total algebraic multiplicity of all eigenvalues of M on U. An $M \in Sp(2n)$is

$$\begin{aligned} &\text{truly hyperbolic}\ , && \text{if } e(M) = 0, \\ &\text{hyperbolic}\ , && \text{if } 1 \in \sigma(M) \text{ and } e(M) = 2, \\ &\text{elliptic}\ , && \text{if } e(M) = 2n, \\ &\text{strongly elliptic}\ , && \text{if } \sigma(M) \subset U \backslash \{1\}. \end{aligned}$$

We denote by $Sp^{th}(2n), Sp^h(2n), Sp^e(2n)$, and $Sp^{se}(2n)$ the subsets of all truly hyperbolic, hyperbolic, elliptic, strongly elliptic matrices in $Sp(2n)$ respectively. $Sp_i^{th}(2n) = \{M \in Sp^{th}(2n) | \alpha(M) = i\}, i = 0, 1$.

We first prove the following lemmas:

Lemma 5.9. The truly hyperbolic matrices $Sp^{th}(4)$ in $Sp(4)$ consist of those symplectic matrix in the open regions I, II, III, V and the curve $B = 4A^2 + 2$ from left to R and then from S to right. Further more, $Sp_1^{th}(4)$ is the region V and $Sp_0^{th}(4) = Sp^{th}(4) \backslash Sp_1^{th}(t)$ is the region I, II, III and the curve $B = 4A^2 + 2$ from left to R and then from S to right. The symplectic matrices in the open region VII as shown in figure 23 are strongly elliptic matrices.

Proof. Let M be a symplectic matrix in the open region I, i.e., the coefficients of its characteristic polynomial $f(\lambda) = \lambda^4 - 4A\lambda^3 + B\lambda^2 - 4A\lambda + 1$ satisfy

$$B > -8A - 2, \qquad B < 4A^2 + 2 \quad \text{and} \quad A < -1. \tag{5.27}$$

Because all signs of the coefficients in the characteristic polynomial are positive, there is no positive real root by Descarte's rule. We can further prove there is no complex roots in the region I. In fact, there are four different negative roots which are also not -1. $\alpha(M) = 0$ so $M \in Sp_0^{th}(4)$. By similar arguments, we complete the proof of this lemma. $\sharp$

Lemma 5.10. The singular set $Sp(4)_1^0$ is the line $B = 8A - 2$. The singular set $Sp(4)_{-1}^0$ is the line $B = -8A - 2$. The singular set $Sp(4)_i^0$ is the line $B = 2$.

Proof. By direct computation. $\sharp$

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Lemma 5.11. The regular set $Sp(4)_1^+$ consists of those symplectic matrix in the regions I, II, III, VI, VII and those curves left to the line $B = 8A - 2$. Or equivalently, $Sp(4)_1^+$ is the left half plane to the line $B = 8A - 2$. $Sp(4)_1^-$ is the right half plane to the line $B = 8A - 2$. Furthermore, if $M \in Sp(4)_1^-$, $\sigma(M) \bigcap (1, \infty) \neq \emptyset$.

Proof. By theorem 2.4.1 in [17], the set $Sp(4)_1^+$ and $Sp(4)_1^-$ are the two connected components in $Sp(4)$. By definition, $Sp(4)_1^+ = \{M | D_1(M) = (-1)^{2-1}(1)^{-2}\det(M - I) = -\det(M - I) < 0\}$. Then $M_2^+ \in Sp(4)_1^+$, $M_2^- \in Sp(4)_1^-$, where

$$M_2^\pm = \begin{bmatrix} \pm 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \pm\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

By the continuity of the map $M \mapsto (A, B)$, the two connected components lies in the two half planes divided by the line $B = 8A - 2$. For M_2^+, the correspondences $A = \frac{tr(M_2^+)}{4} = \frac{5}{4}$, $B = \frac{1}{2}((tr(M_2^+))^2 - tr(M_2^+)^2) = \frac{33}{4}$, then $B > 8A - 2$ and $B = 4A^2 + 2$ which implies that the top half plane containing M_2^+ belongs to $Sp(4)_1^+$. ♯

Theorem 5.7. *For any $\gamma \in \mathcal{P}_{\tau,1}^*(4)$, the index $i_1(\gamma)$ is odd if and only if $\gamma(\tau) \in Sp(4)_1^-$. Equivalently, the index $i_1(\gamma)$ is even if and only if $\gamma(\tau) \in Sp(4)_1^+$.*

Proof. $\gamma(\tau) \in Sp(4)_1^-$, if and only if $\gamma(\tau)$ can be connected to M_2^- without crossing the singular set $Sp(4)_1^0$ (or the line $B = 8A - 2$), if and only if the index $i_1(\gamma)$ is odd, by the equation (5.24) and the definition of $i_1(\gamma)$. ♯

Theorem 5.8. *Suppose x is a τ-periodic solution of the Hamiltonian system and γ_x is the associated symplectic path of x. If $\gamma_x \in \mathcal{P}_{\tau,1}^*(4)$ and $i_1(\gamma_x)$ is an odd integer, then the periodic solution x is linear unstable.*

Proof. Because $\gamma_x \in \mathcal{P}_{\tau,1}^*(4)$ and $i_1(\gamma_x)$ is an odd integer, $\gamma_x(\tau) \in Sp(4)_1^-$. Therefore, $\gamma_x(\tau)$ has an positive real eigenvalue which is larger than 1. By lemma 5.4 and lemma 5.11, the periodic

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solution x is linear unstable. $\sharp$

The proof of lemma 5.12 and theorem 5.9 is similar to the proof of lemma 5.11 and theorem 5.8.

Lemma 5.12. The regular set $Sp(4)^{+}_{-1}$ consists of those symplectic matrix in the regions I, II, III, IV, VII and those curves right to the line $B = -8A-2$. Or equivalently, $Sp(4)^{+}_{-1}$ is the right half plane to the line $B = -8A-2$. $Sp(4)^{-}_{-1}$ is the left half plane to the line $B = -8A-2$. Furthermore, if $M \in Sp(4)^{-}_{-1}$, then $\sigma(M) \bigcap (-\infty, -1) \neq \emptyset$.

Theorem 5.9. *Suppose x is a $\tau-$periodic solution of the Hamiltonian system and γ_x is the associated symplectic path of x. If $\gamma_x \in \mathcal{P}^*_{\tau,-1}(4)$ and $i_{-1}(\gamma_x)$ is an odd integer, then the periodic solution x is linear unstable.*

Theorem 5.10. *Suppose x is a $\tau-$periodic solution of the Hamiltonian system and γ_x is the associated symplectic path of x. If $\gamma_x \in \mathcal{P}^*_{\tau,1}(4) \bigcap \mathcal{P}^*_{\tau,i}(4) \bigcap \mathcal{P}^*_{\tau,-1}(4)$ and $i_1(\gamma_x)$ is an even integer, $i_{-1}(\gamma_x)$ is an even integer and $i_i(\gamma_x)$ is an odd integer, then the periodic solution x is linearly stable.*

Proof. Because $\gamma_x \in \mathcal{P}^*_{\tau,1}(4) \bigcap \mathcal{P}^*_{\tau,i}(4) \bigcap \mathcal{P}^*_{\tau,-1}(4)$, then $\sigma(\gamma_x(\tau)) \bigcap \{1, -1, i\} = \emptyset$. That $i_1(\gamma_x)$ is an even integer implies $\gamma_x(\tau)$ is in the right half plane to the line $B = 8A-2$. That $i_{-1}(\gamma_x)$ is an even integer implies $\gamma_x(\tau)$ is in the left half plane to the line $B = -8A-2$. That $i_i(\gamma_x)$ is an odd integer implies $\gamma_x(\tau)$ is in the lower half plane to the line $B = 2$. Therefore, $\gamma_x(\tau)$ must fall into the region VII. $\sigma(\gamma_x(\tau)) \subseteq U$ and $\gamma_x(\tau)$ is diagonalizable. By lemma 5.4, the periodic solution x is linearly stable. $\sharp$

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